Overcoming the Coordination Problem: Dynamic Formation of Networks[*]

by

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Abstract

We analyze a multi-period entry game among privately informed agents who differ with respect to the number of agents who must enter in order for their own entry to be profitable. In each period agents who have not yet joined decide whether to subscribe to a network. There exists a unique equilibrium that approximates any symmetric equilibrium arbitrarily closely as the discount factor approaches one. This resolves the coordination problem. Ex-post efficiency is necessarily achieved asymptotically as the population size grows large. These results do not hold if subscribers can reverse their decisions without cost. (*JEL* Classification Codes: D82, D85)

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1. Introduction

Adoption/network externalities arise when complementarities exist across agents in the consumption of certain goods or services. Examples include commodities designed for joint consumption or sharing (telephony and data networks), those with indirect scale economies for complementary goods (hardware-software and durable-good servicing), and adoption of innovations and standards where compatibility is valuable.

Due to strategic complementarity, there typically exist multiple, Pareto ranked equilibria in such markets. The worst is a null equilibrium in which no one adopts because no one is ever anticipated to adopt; while at the other end is a "maximum" equilibrium in which a "maximal set of agents" who would adopt when that is what everyone expects to occur, indeed adopt. There may be other equilibria intermediate between these two, sustained by various self-fulfilling expectations. With no outside force present, the particular equilibrium to be realized is indeterminate. This is a well-known coordination problem.

One strand of research has studied inducement schemes as a device to overcome the likelihood of coordination failure in static, simultaneous move entry games. These schemes provide insurance against low adoption or entry rates. Such insurance warrants a sufficient rate of adoption by those who have a low cost of entry, which, in turn, will induce others with higher entry costs to also enter. Dybvig and Spatt [9] and Park [19] devise insurance schemes that will induce certain target equilibria as the unique (symmetric) equilibrium at the minimal expected cost of insurance subsidy. Bagnoli and Lipman [2] study a refund mechanism to induce private contribution to a public project where a sufficient number of people must contribute before the project produces any benefit.

In this paper we analyze the effect of a dynamic adoption process on resolving the coordination problem in the market entry game when agent types are privately and independently drawn from a commonly known distribution. A dynamic adoption process, however, introduces a strategic consideration that is absent in the static game. Individuals who chose to enter early may influence the entry decisions of others who have not yet entered. This creates the possibility that early entrants may launch a domino chain reaction of widespread adoption. However, agents considering early entry will be so motivated only if they expect such a domino chain. Such a domino chain itself relies on a nested sequence of optimistic beliefs of future adopters. At first sight, therefore, it appears that the basic intuition of coordination failure due to multiplicity of self-confirming expectation would continue to prevail in dynamic adoption process. Rather surprisingly, we establish that this is not the case. Specifically, we show that there exists a unique perfect Bayesian equilibrium that approximates any symmetric equilibrium arbitrarily closely as the discount factor approaches one.

In our model, as will be formally described in Section 2, agents' types are ordered by the utility levels agents derive from being a member of the network. Since each member's utility increases as the network gets larger, the higher is the utility an agent derives from the network the lower is the threshold network size for this agent to join profitably. Hence, we say an agent who derives a *higher* utility level from the network has a *lower* type.

First, as a benchmark we analyze the case that agents do not discount the future. Since the timing of action does not matter so long as the final outcome is the same in this case, the exact timing of entry by agents of various types is not pinned down in equilibrium (although the order of entry is). Given that our main interest is on the cases of small yet positive discounting and the agents prefer bringing forward the entry process in these cases, we temporarily impose a stopping rule that if no one entered in some period then no further entry may take place subsequently (this rule makes delaying entry unattractive).

In such equilibria, all agents choose a cutoff strategy in which an agent enters in any period k precisely when his type is no higher than a cutoff level for that period. The cutoff levels are strictly above the lower bound, so entry always occurs with a positive probability. Intuitively, higher types who would need large numbers of other agents also to be in the network in order to find their own entry profitable, will enter later than lower types who would need smaller numbers of other agents in the network to find it profitable. Therefore, an agent of a particular type who has not yet entered can use the common knowledge of the state of the game in period k to determine the expected number of additional entrants, conditional on his own entry in period k and the number of prior entrants. The ability to form these expectations of future entrants generates a backward induction process that uniquely pins down the equilibrium adoption process from the point where all but one agent have entered already all the way back to the point at which no one has entered. The resulting unique equilibrium is in a Markov strategy: the cutoff level in each period is determined entirely by the number of agents who have entered by then.

However, when agents do discount, the equilibrium cutoff levels depend on further details of history such as how many entered in which period. Nevertheless, the basic logic of our argument applies and we show that (i) there is a unique symmetric equilibrium that satisfies the aforementioned stopping rule if the agents are sufficiently patient, and furthermore, (ii) all symmetric equilibria (i.e., even those, if exist, that fail the stopping rule) converge to the unique, no-discount equilibrium as the discount factor tends to one.

The equilibrium adoption process outlined above is fully characterized in Section 3. In a nutshell, this process is illustrated rather well by the typical process of standing ovation in theater performances, where the most impressed/enthusiastic members of the audience initiate the applause, which lowers the threshold for the remaining audience to stand up as well, inducing slightly less impressed members to join the ovation. This process repeats itself to add progressively less impressed members to the ovation, until the accumulated ovation size falls short of persuading any of the remaining audience to stand up.[1]

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[1] We thank Glen Weyl for mentioning this example to us.

Since every adoption along the equilibrium path is optimal relative to the *expectation* of follow-on adoptions, the actual network finally formed, which depends on the realized type-profile, may not be ex post efficient (as elaborated in discussing Farrell and Saloner [10] below). As the number of agents increases, however, the law of large numbers implies that the expected value of the follow-on adoptions becomes ever more reliable an estimate of the actual value and consequently, the ex ante efficiency would converge to the ex post efficiency. Indeed, we show in Section 4 that as the population size increases without bound, virtually all agents who would ever adopt will adopt in the initial period, establishing asymptotic ex post efficiency of the equilibrium outcomes. It may be worth stressing that this result is obtained without restricting attention to symmetric equilibria. We believe, therefore, that existence of asymmetric equilibria in some environments does not undermine our main message that the dynamic adoption process helps resolve the coordination problem.[2]

The aforementioned departure from ex post efficiency arises potentially because adoption decisions are irreversible. In Section 5 we analyze the alternative case in which agents may leave the network, reversing their decisions. The findings are two-fold: The ex post efficiency can indeed be achieved in one set of equilibria, but the equilibrium is no longer unique. In particular, another equilibrium always exists in which the beliefs that no one will ever adopt are self-fulfilling. Section 6 concludes with a summary.

Related Literature

Several authors have studied games with strategic complementarities in dynamic environments in which either every agent's type, or the *actual* distribution of agent types, is common knowledge. Rohlfs [20] considers introductory pricing in his classic paper on telecommunication markets. He analyzes the implications of dis-equilibrium adjustment processes rather than providing an equilibrium analysis of a dynamic game. In his model there remain a multiplicity of equilibria. Gale [13] characterizes the subgame perfect equilibria in a class of monotone games. Monotone games are games of complete information, formed by the indefinite repetition of a stage game with strategic complementarities. In a monotone game once an action has been taken in one stage game it cannot be reversed in any later stage game. Examples of such games include the dynamic entry game analyzed by Gale [12] and the dynamic voluntary contribution games analyzed by Admati and Perry [1] and Marx and Matthews [16]. These monotone games do not resolve the coordination problem associated with the stage game. As in Gale [12], for instance, if entry produces a

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[2] Of course, it is possible that there are sources of aggregate uncertainty that we do not model here yet would provide motivation for some delay even in large markets, including uncertainty with respect to the current state and/or future development of technologies, e.g., "penguin effect" by Choi [6]. Furthermore, our findings are predicated upon the assumption of common knowledge of the law of large numbers. Lack of this common knowledge in the real world may also generate delays and coordination failures even in large markets.

flow of benefits from the time of entry where the benefit flow at any time depends on the proportion of players who have entered by then, but decisions can be taken only at discrete points in time, then there are a multiplicity of subgame perfect equilibria that differ in the length of delay after which everyone enters. The length of delay sustainable in such equilibria gets longer as there are more players in the game. Indeed, when the number of players is sufficiently large there exists a subgame perfect equilibrium in which no one ever enters.[3]

We are interested in an entry game with strategic complementarities in which agents' types are randomly determined and privately known. If there were common knowledge both that the types are positively correlated and of the nature of the correlation, then the theory of global games developed by Carlsson and van Damme [4] would apply. Morris and Shin [17] show that even when there is only a small amount of heterogeneity in types in such games there will often be a unique equilibrium.[4] The common knowledge of the way in which beliefs are correlated allows individuals, through iterated elimination of strictly dominated strategies, to condition their beliefs as to how others will act on the knowledge of their own individual types. Like global games, uncertain heterogeneity underlies the logic of induction that resolves the coordination problem in our setting. However, without commonly known correlation of types, the theory of global games does not apply to the kind of entry games we consider and thus, the core argument is different.

Bliss and Nalebuff [3] analyze a dynamic game of incomplete information. They analyze a public good game in which provision of the good requires a costly action be taken by only one of N players. The cost of that action to any player is randomly drawn from a commonly known distribution of possible costs and is private information. The gross benefit of provision to each of the N players, including the player who provides the good is a declining function of the time that elapses before the good is provided. Only the player who provides the good incurs a cost. Once a player provides the good the game is over. There is a unique symmetric Bayesian equilibrium to this game. In this equilibrium the strategy of a player with a given cost of provision is characterized by an optimal amount of time to allow to elapse before offering to provide the good because the higher is a player's cost of provision the less he has to lose by waiting to see if there is some other player who will act first because that other player has a lower cost of provision

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[3] In Gale's entry game there is a critical proportion of players who must enter before it is profitable to enter. As the number of players, N, increases, the critical number of players who must enter for it to be profitable for any player to enter, n^*, also increases without bound. There is a subgame perfect equilibrium in which everyone enters after a delay of $n^* - 1$ periods (Gale [12], Theorem 2). If N is sufficiently large (hence, so is the equilibrium delay), it is possible that no one would enter in any period supported by the off-equilibrium beliefs that any entry will be followed by an equilibrium of the subgame with long enough a delay so that the entrant would not recover the loss from being a sole entrant (Corollary, p.9).

[4] Various dynamic versions of the global games have been studied, producing *inter alia* additional insights on the conditions for equilibrium uniqueness, e.g., by Chamley [5], Heidhues and Melissas [14], Hellwig, Mukherji and Tsyvinski [15], and Dasgupta [7].

(a higher opportunity cost of delay). The first player to move reveals his type. In the Bliss and Nalebuff model everything is resolved once one player has entered. This simplifies the characterization of the symmetric equilibrium strategy. However, in the market entry game we consider the game may not be over at time t as long as some players have not yet entered. Consequently, the payoff to a player who enters at any time t depends upon not only how many players have entered by t but also how many other players will enter after t and the time path of the sequence of those entrants. This additional strategic consideration arises from the general strategic complementarities that do not exist in Bliss and Nalebuff.

Farrell and Saloner [10] provide an analysis of a dynamic entry game with strategic complementarities in an incomplete information environment. To focus on the effects of communication on the likelihood of coordination failure, which is their main interest, they analyze the game with only two players and no discounting (hence, no cost of delay). They show that there is a unique, perfect Bayesian equilibrium in the game they consider. Consequently, the coordination problem associated with a multiplicity of equilibria in the stage game is resolved. While multiplicity of equilibrium is not a problem, there is still a positive probability of ex post "coordination failure." The equilibrium is characterized by a partition of types into three prime groups: "leader" types who start the adoption process by entering in the initial period, "follower" types who will enter in the next period if and only if the other player entered already, and the rest of types who will never enter. (The leader types are partitioned into two subgroups, the types in one will find it profitable to have entered, ex post, only if both players enter, and those in the other will find it profitable regardless.) Consequently, if both players happen to be followers they will both fail to enter even though, ex post, they would have preferred that both enter. Farrell and Saloner call this type of coordination failure "excess inertia." They show that if the game is modified by adding an initial period in which each player can announce whether or not they want to enter this will change the equilibrium in the subsequent game by reducing the size of leader types, making this kind of coordination failure more likely. Another form of ex post coordination failure arises when a "leader" regrets ex post because the other agent did not enter. This is termed "excess momentum" by Farrell and Saloner. Since communication reduces the set of leader types communication reduces the likelihood of the second kind of ex post coordination failure.[5]

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[5] In a related environment of standard adoption (but without private information), Ostrovsky and Schwarz [18] investigate the impact of the presence of noise in compliance times and characterize when a superior standard may be adopted and when it may not. In another standard adoption environment where the "battle of the sexes" is the base game, Farrell and Saloner [11] analyze and compare three mechanisms for achieving coordination: when preplay communication is allowed until agreement is reached or time runs out, when unilateral adoption is possible in each non-terminal period to preempt coordination on the opponent's preferred outcome, and when both are possible. No private information exists in both of these studies, hence the nature of problem is different from ours. Dixit [8] also presents a complete information game of standard adoption, in the unique subgame perfect equilibrium of which the socially

We study an entry game with incomplete information that extends that of Farrell and Saloner [10] in three main directions. First, we generalize their analysis to a game with any finite number of agents. Second, we also characterize the equilibrium when entrants derive utility each period and discount future flows so that the dynamics, as well as the final network formed, are of importance. Third, we also analyze the case that agents can reverse their decision at no cost. For our purpose of investigating the effect of dynamic adoption on coordination, these extensions allow richer dynamic interactions in more realistic settings, allowing potentially more robust analytic findings and new insights for large population.

In an unpublished paper developed independently[6], Xue studies a dynamic coordination game similar to ours but with special features that are not present in our own model, namely, the benefit from network is not realized unless unanimously adopted and the type enters in the utility function linearly. Thus, the two studies are differentiated both in their applicability (momentum building with binary outcomes such as in political uprising vs. gradual adoption of goods with network externalities) and the specific analytical approaches taken,[7] even though the results for no discounting case are similar.

2. Model

There are N ex ante identical agents, indexed by $i \in \mathcal{I} = \{1, \cdots, N\}$, who are privately informed of their own *types* $t \geq 0$ which are independent draws from a common distribution function $F : \Re_+ \to [0,1]$. Let $f : \Re_+ \to \Re_+$ denote the corresponding density function. By rescaling the types if necessary, we let $0 = \inf(\text{supp}(F))$ so that $F(t) > 0$ for all $t > 0$. We assume that F is continuous with $F(0) = 0$ (i.e., F is nonatomic) and f is continuous and bounded. For expositional convenience only, we assume $F(t) < 1$ for all $t \geq 0$.

There are infinite periods indexed by $k = 1, 2, \cdots$. At the beginning of each period k the number n_{k-1} of agents who adopted/subscribed up until period $k-1$ is common knowledge; Based on the public history $h_k := (n_1, \cdots, n_{k-1})$ the agents who have not adopted already simultaneously choose either to adopt the network product or not. We let $h_1 = \emptyset$. Once adopted, agents cannot reverse their choices in future periods.

An agent who adopts in period k' derives a stage utility from the network product in every period $k \geq k'$, determined by his type t and the network size in period k measured by the number of all adopters, n_k: A t-type agent derives a utility of $u_t(n_k) \in \Re$ in period k. The stage utility to a non-adopter is normalized to $u_\phi = 0$. Each agent's objective is to maximize the expected δ-discounted average of utility stream with a discount factor $\delta \leq 1$:

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inferior standard gets adopted by everyone.

[6] J. Xue, Collective Behavior with Endogenous Thresholds, mimeo, University of Cambridge, 2004.

[7] In particular, in the general case that there is discounting, equilibrium is not guaranteed to be in cutoff strategies in our setting, unlike in Xue's paper. This is because the net benefit of adopting as opposed to waiting in a certain period may not be monotonic in the agent's type when the agent's utility is not linear in his type, and is affected by how the network evolves as well as its final size. Not relying on the cutoff nature of equilibrium, therefore, our general approach is different (see Proof of Theorem 2 (b)).

That is, each agent maximizes the expected value of

$$(1-\delta)\sum_{k=1}^{\infty}\delta^{k-1}\mathrm{u}_k \tag{1}$$

if $\delta < 1$, where u_k is the utility in period k, which is 0 if the agent has not adopted yet and is $u_t(n_k)$ if the agent of type t has adopted; and maximizes the limit of the expected value of (1) as $\delta \to 1$ if the discount factor is $\delta = 1$.

An agent's type, t, measures how reluctant he is to join the network, so a higher type means a more conservative agent who needs a larger network to benefit by joining. Hence, we assume that $u_t(n)$ is strictly increasing in $n = 1, \cdots, N$, strictly decreasing and continuous in t, and that

$$u_0(1) = 0 \quad \text{and} \quad \exists\, \bar{t} \;\; s.t. \;\; u_{\bar{t}}(N) = 0. \tag{2}$$

The first equality, that the most "enthusiastic" type is indifferent between being a sole member of the network and being a non-member, is for expositional convenience.[8] Clearly, $\bar{t} > 0$ defined above is unique because $u_t(N)$ strictly decreases in t and $u_0(N) > u_0(1) = 0$. We denote this game by Γ.

An agent i's *period-k strategy* when he has not adopted yet, given a history $h_k = (n_1, \cdots, n_{k-1})$, is an integrable function that maps types to adoption probabilities, i.e.,

$$a^i(\cdot|h_k) \,:\, \Re_+ \to [0,1]$$

where $a^i(t|h_k)$ is the probability that the agent i adopts (the network product) when his type is t, if he has not adopted up to the previous period. A function $a^i(\cdot|h_k)$ is a *cutoff strategy at (a cutoff level)* $\hat{t} \geq 0$, if $a^i(t|h_k) = 1$ for all $t < \hat{t}$ and $a^i(t|h_k) = 0$ for all $t > \hat{t}$. An agent i's *strategy* is a collection $\{a^i(\cdot|h_k)\}$ for all possible h_k, which we denote by a^i as shorthand. A strategy a^i is a *cutoff strategy* if $a^i(\cdot|h_k)$ is a cutoff strategy for every possible h_k.

Definition: *A strategy profile* $(a^i)_{i\in\mathcal{I}}$ *is a (perfect Bayesian) equilibrium of* Γ *if each agent i's period-k strategy after each possible* h_k *is a best response to* $(a^j)_{j\neq i}$ *conditional on* h_k.

Note that we suppress the belief profile in defining equilibrium for ease of exposition: In each period the belief on the type of agents who have not yet adopted is updated by Bayes rule along the equilibrium path; In the case that someone adopted when no

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[8] All the main results of this paper hold when $u_0(1)$ assumes any other value subject to one caveat: If $u_0(1)$ is a sufficiently large negative number for given N, the value of τ_N in Theorem 1 may be negative, in which case no adoption occurs in the *unique* equilibrium. Note that even in this case the coordination problem does not exist in the sense of multiple (Pareto-ranked) equilibria. Furthermore, adoption takes place in the unique equilibrium as N increases and ex post efficiency obtains asymptotically by the same reasoning behind Theorem 4. As a separate point, observe also that adoption is not a dominant action for any type if $u_0(1) < 0$ and hence, the logic behind the global games does not apply.

one is supposed to adopt, which is the only deviation detectable by other players in our context, the belief on the type of non-adopters/non-deviators remain the same and the deviator/adopter's type is inconsequential because it does not affect other agents' utility.

3. Unique Symmetric Equilibrium

In this section we focus our attention on symmetric equilibrium and show that there exists a unique symmetric equilibrium of Γ when δ is sufficiently large. First, we construct the unique symmetric equilibrium for $\delta = 1$ and show that it is a cutoff equilibrium and the cutoff level in each period depends only on the total number of agents who already adopted (Theorem 1). Then, we show that this equilibrium approximates arbitrarily closely any symmetric equilibria there may be as δ tends to one (Theorem 2). For expositional ease, an equilibrium means a symmetric equilibrium unless indicated otherwise.

Irreversibility of adoption is essential for the uniqueness result. Although irreversibility is a strong assumption in some circumstances, there are situations in which it is plausible. For instance, if adoption requires a big sunk cost upfront,[9] once adopted there is no benefit of reversing the decision even if the future stream of surplus is not likely to recoup the sunk cost. In addition, irreversibility renders off-equilibrium beliefs inconsequential as mentioned above and thus, suppresses tricky questions concerning which beliefs would be sensible and how they would come to be common knowledge.

From now until Theorem 1, we analyze the case that $\delta = 1$. Due to no discounting, the agent's objective amounts to maximizing the "terminal" stage utility level that will prevail after the adoption process has ended. The observation that agents only care about the final network size of any adoption process simplifies the analysis for the case $\delta = 1$ because the details of the adoption process leading to the final network can be ignored. However, it allows an inessential indifference of an agent between adopting now and adopting later so long as the final network will be the same. For example, if all but one agent already adopted the remaining agent is indifferent between adopting now and adopting in any later period because his payoff, defined as the limit of (1) as $\delta \to 1$, would be $u_t(N)$ regardless of when he adopts. To bypass this problem, we temporarily impose a stopping rule that

(sr) if no one adopted the network in some period k then no further adoption is allowed,

so that only those agents who adopted by then benefit from the network in future periods. Later, in Theorem 2, we remove this stopping rule for $\delta < 1$.

To characterize equilibrium for the case $\delta = 1$, we need to characterize the equilibrium of the continuation game following every possible history, i.e., any $h_k = (n_1, \cdots, n_{k-1})$ such that $0 < n_1 < n_2 < \cdots < n_{k-1} < N$, as well as the null history $h_1 = \emptyset$. As will become clear in the analysis, what matters in the strategic decisions in the continuation game

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[9] Our model treats such a cost, C, as spread evenly across all period, i.e., $u_t(n)$ is lower by $-\frac{C}{1-\delta}$.

following h_k is the total number n_{k-1} of adopters by then (equivalently, the number of agents who have not adopted), not how it evolved. So, we define the *state (variable)* s for a period k with a history $h_k = (n_1, \cdots, n_{k-1})$ as $s = N - n_{k-1}$, i.e., the number of non-adopters after h_k, who we refer to as the "remaining" agents.

Given any equilibrium of Γ, consider the continuation game starting from an arbitrary period of state $\tilde{s}$ along the equilibrium path. We denote this continuation game by $\Gamma_{\tilde{s}}(g)$ where g is the posterior density of the remaining agent's types at the beginning of the continuation game. Since agents of any type greater than $\bar{t}$ would never adopt,

$$g|_{t \geq \bar{t}} = f|_{t \geq \bar{t}}, \;\; G(\bar{t}) \leq F(\bar{t}) < 1, \;\; g(t) \leq \frac{f(t)}{1 - F(\bar{t})} \; \forall t \geq 0, \text{ and } G \text{ is nonatomic,} \qquad (3)$$

where G is the cdf associated with g.

Below we characterize equilibrium of all continuation games $\Gamma_{\tilde{s}}(g)$ with any g that satisfies (3) via induction on $\tilde{s} = 1, \cdots, N$, which will establish the unique equilibrium characterized in Theorem 1. Since the induction process is rather involved, we provide a brief outline. First, we show in Step 1 that if only one agent remains, i.e., in $\Gamma_1(g)$, he would adopt if and only if his type is below $\bar{t}$. Next, we show in Step 2 that if two agents remain, i.e., in $\Gamma_2(g)$, the unique continuation equilibrium is a cutoff equilibrium that only depends on the type distribution g. In doing so, we first observe that the same final network (grand network) would be reached regardless of whether a particular agent adopts in the current period or in the next period, if the other remaining agent were to adopt in the current period. Then, since adoption paths do not matter if they lead to the same final network because $\delta = 1$, we determine the equilibrium cutoff type by equating the expected payoffs from adopting and not in this period, conditional on the other agent not adopting in this period. This equation turns out to have a unique solution.

Finally, we present in Step 3 a general induction argument showing that if all continuation games $\Gamma_{\tilde{s}}(g)$, $\tilde{s} < \tilde{r}$, have a unique equilibrium that only depends on g, so does $\Gamma_{\tilde{r}}(g)$. One potential source of problem in this step is the possibility that the cutoff levels in future periods of the same state $\tilde{s}$ ($< \tilde{r}$) may differ if they followed different adoption history because then the posterior type distribution of the remaining agents (g) may differ. We show that this cannot happen (Lemma 3). Then, in conjunction with the observation that the same final network would be reached whether a particular agent adopts in the current period or in the next period so long as some other agent(s) were to adopt in the current period (Lemma 4), we uniquely determine the equilibrium cutoff type by equating the expected payoffs from adopting and not in this period, conditional on no other agent adopting in this period (Lemma 5). This leads to the conclusion of the induction argument.

STEP 1: Consider $\Gamma_1(g)$, the continuation game when a state $s = 1$ has been reached, i.e., only one agent remains in some period k and the posterior belief on his type is g. It is trivial that this last agent will adopt precisely when his type does not exceed $\bar{t}$ defined in (2).

Lemma 1: *The equilibrium strategy in any continuation game $\Gamma_1(g)$ is a cutoff strategy at $\tau_1(g) \equiv \bar{t}$.*

STEP 2: Consider a continuation game $\Gamma_2(g)$. Consider one remaining agent, say i, of type $t_i \le \tau_1$. If the other remaining agent, say j, were to adopt in this period, agent i would get a utility of $u_{t_i}(N)$ by adopting in this period; if agent i waited in this period he would adopt in the next period by Lemma 1 (because $t_i \le \tau_1$), hence again get a utility of $u_{t_i}(N)$ eventually. Therefore, agent i's optimal decision in this period depends on what would happen in the contingency that agent j were to not adopt in this period. In this contingency, agent i would get a utility $u_\phi = 0$ by not adopting in this period because no further adoption would ensue due to the postulated stopping rule; if agent i adopted in this period, he would get $u_{t_i}(N)$ eventually in case agent j joins in the next period and $u_{t_i}(N-1)$ otherwise. Since agent j's response in the next period is independent of t_i, the expected utility of agent i from adopting decreases in t_i, whereas that from waiting is 0. Consequently, agent i should employ a cutoff strategy at a level, say $\hat{t}$, which must be the strategy of agent j as well since we focus on symmetric equilibrium. Note that agent i strictly prefers waiting in this period if his type is sufficiently close to τ_1, hence $\hat{t} < \tau_1$.

The condition that characterizes $\hat{t}$ is the following: agent i of $\hat{t}$-type is indifferent between adopting and waiting in this period given that agent j follows a cutoff strategy at $\hat{t}$ in this period and a cutoff strategy at τ_1 in state $s = 1$, i.e.,

$$\Big(u_{\hat{t}}(N) \int_{\hat{t}}^{\tau_1} g(t)dt + u_{\hat{t}}(N-1) \int_{\tau_1}^{\infty} g(t)dt\Big) \Big/ \int_{\hat{t}}^{\infty} g(t)dt \;=\; 0. \tag{4}$$

The left hand side, LHS, of (4) is the expected utility of a $\hat{t}$-type agent when he adopts in the current period conditional on the other remaining agent waits, while the RHS is that when he waits in the current period. Note that the LHS of (4) is strictly decreasing in $\hat{t}$, clearly from a positive value when $\hat{t} = 0$ to a negative value when $\hat{t} = \tau_1$. Hence, there exists a unique value of $\hat{t}$ that solves (4), which is the equilibrium cutoff level in the first period of $\Gamma_2(g)$, denoted by $\tau_2(g)$. Let $[g]_2$ denote the set of density functions g' that satisfy (3) and coincide with g when restricted to $t \ge \tau_2(g)$, i.e., $g'|_{t \ge \tau_2(g)} = g|_{t \ge \tau_2(g)}$. Clearly, the solution to (4) does not change when g is replaced by any $g' \in [g]_2$. Together with Lemma 1, we have

Lemma 2: *The unique symmetric equilibrium of a continuation game $\Gamma_2(g)$ is a cutoff strategy equilibrium with cutoff levels at $\tau_2(g)$, the unique level of $\hat{t}$ that solves (4), when $s = 2$ and at $\tau_1(g) = \bar{t}$ when $s = 1$. Furthermore, the continuation games $\Gamma_2(g)$ and $\Gamma_2(g')$ have the same unique equilibrium if $g' \in [g]_2$.*

Note that τ_1 is independent of any other cutoff levels and thus, the other cutoff level in $\Gamma_2(g)$, $\tau_2(g)$, is uniquely determined as the solution to a monotone function as explained above. If there are more than two agents, however, the cutoff levels for $s > 1$ may depend on one another. Due to this possibility of interdependence, the task of checking uniqueness is much more complex,[10] especially when $\delta < 1$ as can be seen in the proof of Theorem 2 in Appendix. The argument is somewhat simpler when $\delta = 1$ because all that matters is the final network size, as presented below.

STEP 3: Fix a state $\tilde{s}$ and consider a continuation game $\Gamma_{\tilde{s}}(g)$. Provided that $\Gamma_{\tilde{s}}(g)$ has a unique cutoff strategy equilibrium with the initial cutoff level $\tau_{\tilde{s}}(g)$, define the equivalence class of density functions

$$[g]_{\tilde{s}} := \{g' \,|\, g' \text{ satisfies (3) and } g'|_{t \geq \tau_{\tilde{s}}(g)} = g|_{t \geq \tau_{\tilde{s}}(g)}\}.$$

Consider the following property:

[A] (i) The unique equilibrium of any continuation game $\Gamma_{\tilde{s}}(g)$ is a cutoff strategy equilibrium with cutoff levels $0 \leq \tau_{\tilde{s}}(g) < \tau_{\tilde{s}-1}(g) < \cdots < \tau_1(g) = \bar{t}$. (ii) If $g' \in [g]_{\tilde{s}}$, the continuation games $\Gamma_{\tilde{s}}(g')$ and $\Gamma_{\tilde{s}}(g)$ have the same unique equilibrium.

Note that this property holds when $\tilde{s} = 1$ by Lemma 1 and when $\tilde{s} = 2$ by Lemma 2. We now make an induction hypothesis that the property [A] holds for all $\tilde{s} < \tilde{r}$ where $\tilde{r} = 3, \cdots, N$. Below we establish that under this hypothesis the property [A] holds for $\tilde{s} = \tilde{r}$ as well.

Fix an equilibrium of an arbitrary continuation game $\Gamma_{\tilde{r}}(g)$ that starts from period k of the original game Γ. Let g^+ denote the posterior density function on the remaining agent's type that prevails in period $k+1$ according to the equilibrium. (Note that g^+ does not depend on how many agents have adopted in period k.) If $j > 0$ agents adopt in period k, a continuation game $\Gamma_{\tilde{r}-j}(g^+)$ ensues in period $k+1$ and the remaining agents adopt according to the cutoff level $\tau_{\tilde{r}-j}(g^+)$ in period $k+1$ by the induction hypothesis [A] (i). If, on the other hand, j agents have adopted between periods k and k' (with some agents adopting in period k'), then the continuation game starting from period $k'+1$ is $\Gamma_{\tilde{r}-j}(g^{+\prime})$ where $g^{+\prime}$ is the posterior as of period $k'+1$. Since cutoff strategies have been followed from period $k+1$, it has to be the case that $g^{+\prime}|_{t \geq \tau_{\tilde{r}-j}(g^{+\prime})} = g^+|_{t \geq \tau_{\tilde{r}-j}(g^{+\prime})}$, i.e., $g^+ \in [g^{+\prime}]_{\tilde{r}-j}$. By [A] (ii), therefore, we have $\tau_{\tilde{r}-j}(g^+) = \tau_{\tilde{r}-j}(g^{+\prime})$. Since this is true for all $j = 1, \cdots, \tilde{r}-1$, and $k' > k$, we have

Lemma 3: *In the continuation game $\Gamma_{\tilde{r}}(g)$ that starts from period k of Γ, if $j = 1, \cdots, \tilde{r}-1$ agents have adopted between periods k and k' $(\geq k)$, the remaining agents*

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[10] It may be worth noting that in a dynamic model of global games, Hörner proves asymptotic uniqueness of equilibrium for two players, but has a counterexample to uniqueness with three players (J. Hörner, Payoff-Dominance in Dynamic Coordination Games, mimeo, Northwestern University, 2004).

adopt according to the cutoff level $\tau_{\tilde{r}-j}(g^+)$ *in period* $k'+1$ *where* g^+ *is the posterior density that prevails in period* $k+1$.

We establish two more lemmas for the continuation game $\Gamma_{\tilde{r}}(g)$.

Lemma 4: *Pick an arbitrary remaining agent* i *in the first period,* k*, of* $\Gamma_{\tilde{r}}(g)$*. Consider any contingency that* $m > 0$ *of the other* $\tilde{r}-1$ *remaining agents were to adopt in period* k *according to the equilibrium strategy of* $\Gamma_{\tilde{r}}(g)$*. Then, the final network size that would realize when the agent* i *adopts in period* k *is the same as the final network size that would realize when he adopts in period* $k+1$.

Proof. In this proof we use τ_s as shorthand for $\tau_s(g^+)$ described in Lemma 3 for $s \leq \tilde{r}-1$. Consider the case that the agent i adopted in period k, so that the state in period $k+1$ is $s_1 = \tilde{r}-m-1 < \tilde{r}$, hence all remaining agents of types lower than the equilibrium cutoff level τ_{s_1} would adopt in period $k+1$ by Lemma 3. Let s_2 be the state of period $k+2$ that arises as a result. If $s_2 = s_1$ then no further adoption comes forth by Lemma 3, in which case s_1 is called the terminal state; otherwise, i.e., if $s_2 < s_1$ then all remaining agents of types lower than the equilibrium cutoff level τ_{s_2} would adopt in period $k+2$, resulting in a state s_3 of period $k+3$. If $s_3 = s_2$ then s_2 is the terminal state; otherwise, the state keeps being updated analogously for subsequent periods. The updating should stop because there are finite states. Denoting the terminal state by s_x, we have a sequence of states $s_1 > s_2 > \cdots > s_x$ and the associated cutoff levels $\tau_{s_1} < \tau_{s_2} < \cdots < \tau_{s_x}$ for periods $k+1, \cdots, k+x$, respectively. Note $s_{x+1} = s_x$ by construction.

Consider the alternative case that the agent i did not adopt in period k, so that the state in period $k+1$ is $s'_1 = \tilde{r}-m = s_1+1$, hence all remaining agents of types lower than the equilibrium cutoff level $\tau_{s'_1}$ would adopt in period $k+1$. Let s'_2 be the state of period $k+2$ that arises as a result. If $s'_2 = s'_1$ then no further adoption comes forth by Lemma 3, hence s'_1 is the terminal state; otherwise, i.e., if $s'_2 < s'_1$ then all remaining agents of types lower than the equilibrium cutoff level $\tau_{s'_2}$ would adopt in period $k+2$, resulting in a state s'_3 of period $k+3$. If $s'_3 = s'_2$ then s'_2 is the terminal state; otherwise, the state keeps being updated analogously. Denoting the terminal state by s'_y, we have a sequence of states $s'_1 > s'_2 > \cdots > s'_y$ and the associated cutoff levels $\tau_{s'_1} < \tau_{s'_2} < \cdots < \tau_{s'_y}$ for periods $k+1, \cdots, k+y$, respectively. Again, $s'_{y+1} = s'_y$ by construction.

The claim of the Lemma is proved if $s_x = s'_y$. In fact, it is easy to see that

$$s_x = s'_y \ \text{ ensues if } \ s_j = s'_\ell \ \text{ for some } \ 1 \leq j \leq x \ \text{ and } \ 1 \leq \ell \leq y, \tag{5}$$

because then $s_{j+1} = s'_{\ell+1}$ and the subsequent updating of the state is the same between the two sequences.

Note that $s'_1 > s'_2$ because the agent i adopts in period $k+1$. Since $s_1 = s'_1 - 1$ by construction as noted earlier, $s_1 \geq s'_2$. If $s_1 = s'_2$ then the claim is proved by (5).

Suppose otherwise, i.e., $s_1 > s'_2$. By construction, $s'_2 = s'_1 - 1 - \#(0, \tau_{s'_1}] = s_1 - \#(0, \tau_{s'_1}]$ where $\#(0, \tau]$ is the number of agents other than i who remain in the period $k+1$ and have types in $(0, \tau]$. Similarly, $s_2 = s_1 - \#(0, \tau_{s_1}]$ by construction. Since $s'_1 > s_1$ implies $\tau_{s'_1} < \tau_{s_1}$, it follows that $\#(0, \tau_{s'_1}] \leq \#(0, \tau_{s_1}]$, hence $s'_2 \geq s_2$.

The claim follows by (5) if $s'_2 = s_2$, hence suppose $s'_2 > s_2$ in the sequel. By construction, $s'_3 = s'_1 - 1 - \#(0, \tau_{s'_2}] = s_1 - \#(0, \tau_{s'_2}]$. Since $s_1 > s'_2$ implies $\tau_{s_1} < \tau_{s'_2}$, it follows that $s_2 \geq s'_3$. Since the claim follows if $s_2 = s'_3$, suppose $s_2 > s'_3$ in the sequel.

Proceeding analogously, we deduce that $s_x = s'_y$ unless $s'_j > s_j > s'_{j+1} > s_{j+1}$ for all $j = 1, 2, \cdots$. However, these inequalities contradict $s_{x+1} = s_x$, an equality that must hold by construction, hence we conclude that $s_x = s'_y$, i.e., the final network sizes are the same. (For later use we note that the logic of this proof does not hinge on symmetry, hence extends straightforwardly to asymmetric equilibria.) ■

Lemma 5: *Fix an equilibrium of an arbitrary continuation game* $\Gamma_{\tilde{r}}(g)$ *that starts from period* k*. The equilibrium strategy in period* k *is a cutoff strategy whose cutoff level, denoted by* $\tau_{\tilde{r}}(g)$*, is uniquely determined by* g*, and* $\bar{t}_{N-\tilde{r}+1} \leq \tau_{\tilde{r}}(g) < \tau_{\tilde{r}-1}(g)$ *where* $\bar{t}_{N-\tilde{r}+1} \geq 0$ *is the unique solution to* $u_t(N - \tilde{r} + 1) = 0$*.*

Proof. In this proof we continue to use τ_s as shorthand for $\tau_s(g^+)$ defined in Lemma 3 for $s \leq \tilde{r} - 1$. Consider an arbitrary remaining agent i in period k. Suppose his type is $t_i \leq \tau_{\tilde{r}-1}(g^+)$. If he waited while $m > 0$ other agents adopted in this period, by adopting in the next period he can induce the same final network size as when he adopted in this period, according to Lemma 4. In fact, he will indeed adopt in the next period by Lemma 3 because $t_i \leq \tau_{\tilde{r}-1}(g^+) \leq \tau_{\tilde{r}-m}(g^+)$. Hence, adopting and waiting are equivalent in this contingency and, therefore, the optimal decision of the agent i in period k is determined by what would happen in the contingency that no agent other than i would adopt in period k. In this latter contingency, if the agent i adopted, then his expected utility is

$$\sum_{j=0}^{\tilde{r}-1} u_{t_i}(N - \tilde{r} + 1 + j)\text{Prob}(j|g^+) \tag{6}$$

where $\text{Prob}(j|g^+)$ is the probability, conditional on agent i being the sole adopter in this period, j more agents adopt eventually. (More precise expressions of (6) and (7), respectively, are (12) and (15) in the Appendix, evaluated at $\delta = 1$.) If the agent i did not adopt, the adoption process would end and he would get $u_\phi = 0$. Again, since other remaining agents' behavior does not depend on t_i, the value of (6) strictly decreases in t_i. Hence, the equilibrium strategy of agent i in period k (hence, that of any other remaining agent by symmetry) is a cutoff strategy at, say $\hat{t}$. The equilibrium level of $\hat{t}$ is characterized by

$$\sum_{j=0}^{\tilde{r}-1} u_{\hat{t}}(N - \tilde{r} + 1 + j)\text{Prob}(j|g|_{t \geq \hat{t}}) = 0 \tag{7}$$

where $\mathrm{Prob}(j|g|_{t\geq\hat{t}})$ is calculated based on the fact that $g^+ = g|_{t\geq\hat{t}}$. If $0 \leq \hat{t} < \hat{t}' \leq \tau_{\tilde{r}-1}(g)$, the distribution represented by $g|_{t\geq\hat{t}'}$ first-order stochastically dominates that represented by $g|_{t\geq\hat{t}}$. Since the equilibrium cutoff levels in the continuation games $\Gamma_{\tilde{r}-1}(g|_{t\geq\hat{t}})$ and $\Gamma_{\tilde{r}-1}(g|_{t\geq\hat{t}'})$ are the same by [A] (ii), therefore, as $\hat{t}$ increases from 0 to $\tau_{\tilde{r}-1}(g)$, the distribution represented by $\mathrm{Prob}(j|g|_{t\geq\hat{t}})$ deteriorates in the sense of first-order stochastic dominance. Together with the fact that $u_{\hat{t}}(\cdot)$ decreases in $\hat{t}$, we deduce that the LHS of (7) strictly decreases in $\hat{t}$, hence there is a unique value of $\hat{t}$ that solves (7), which is $\tau_{\tilde{r}}(g)$. Clearly, $\tau_{\tilde{r}}(g) \geq \bar{t}_{N-\tilde{r}+1}$ because the LHS of (7) is nonnegative when $\hat{t} = \bar{t}_{N-\tilde{r}+1}$. Consider a $\tau_{\tilde{r}-1}(g)$-type agent: his expected utility would be 0 if he already adopted and $\tilde{r}-2$ agents remain whose type is distributed according to $g|_{t\geq\tau_{\tilde{r}-1}(g)}$. So, his expected utility would be negative if $\tilde{r}-1$ agents remain with the same type distribution. This means that the LHS of (7) is negative at $\hat{t} = \tau_{\tilde{r}-1}(g)$ and, therefore, $\bar{t}_{N-\tilde{r}+1} \leq \tau_{\tilde{r}}(g) < \tau_{\tilde{r}-1}(g)$. (For later use we note that this proof does not hinge on symmetry, hence extends straightforwardly to asymmetric equilibria.) ∎

Recall the induction hypothesis that the property [A] holds for all $\tilde{s} < \tilde{r}$ where $\tilde{r} = 3, \cdots, N$. We now verify that the property [A] holds for $\tilde{s} = \tilde{r}$ as well. First, Lemmas 3 and 5 establish that there is a unique equilibrium for any continuation game $\Gamma_{\tilde{r}}(g)$, which is a cutoff strategy equilibrium with cutoff levels at $\tau_{\tilde{r}}(g)$ in the initial period and at $\tau_s(g|_{t\geq\tau_{\tilde{r}}(g)})$ in any subsequent period of state $s < \tilde{r}$. Since $\tau_{\tilde{r}}(g) < \tau_s(g)$ for all $s < \tilde{r}$ as implied by Lemma 5 and the hypothesis [A] (i), it follows from [A] (ii) that $\tau_s(g|_{t\geq\tau_{\tilde{r}}(g)}) = \tau_s(g)$ for all $s < \tilde{r}$. Thus, we have established the property [A] (i) for $\tilde{s} = \tilde{r}$.

Next, consider g and g' such that $g'|_{t\geq\tau_{\tilde{r}}(g)} = g|_{t\geq\tau_{\tilde{r}}(g)}$. Notice that the LHS of (7) evaluated at $\hat{t} = \tau_{\tilde{r}}(g)$, which is 0 by definition of $\tau_{\tilde{r}}(g)$, does not change when g is replaced with g'. Therefore, $\tau_{\tilde{r}}(g') = \tau_{\tilde{r}}(g)$. Consequently, the equilibrium cutoff levels for periods of state $s < \tilde{r}$ in the continuation games $\Gamma_{\tilde{r}}(g)$ and $\Gamma_{\tilde{r}}(g')$ are identical by the induction hypothesis [A] (i). This establishes the property [A] (ii) for $\tilde{s} = \tilde{r}$, completing the induction process that leads to the conclusion that [A] holds for all $\tilde{s} = 1, \cdots, N$.

Since $\tau_{N-1}(f) > 0$ from [A] applied to $\tilde{s} = N$ and $F(t) > 0$ for all $t > 0$, the LHS of (7) is strictly positive when $g = f$, $\tilde{r} = N$ and $\hat{t} = 0$. This means that $\tau_N(f) > 0$. Therefore, we have proved the following characterization of the equilibrium of Γ, where we use τ_s as shorthand for $\tau_s(f)$, a notation that we adopt in the sequel.

Theorem 1: *If $\delta = 1$, there exists a unique symmetric equilibrium of Γ. In this equilibrium, the remaining agents' strategy after any history is a cutoff strategy at a level that depends only on the state s (i.e., the number of remaining agents), denoted by τ_s, and $0 < \tau_N < \tau_{N-1} < \cdots < \tau_1 = \bar{t}$.*

Observe that there are many inessential variations of this equilibrium when the stopping rule is lifted: All agents of types below τ_s adopt in the first $k_s > 0$ periods from the state having reached s.

Some details of the analysis up to now relies on the fact $\delta = 1$, hence is not readily applicable to the case $\delta < 1$. If $\delta < 1$, an agent would prefer adopting earlier rather than later if adopting later delays the adoption process even though it leads to the same network eventually. In evaluating the benefit of adopting as opposed to waiting, therefore, the time paths following adoption by some other agents come into the equation even if the final network will be the same regardless of whether the agent in question adopts now or in the next period, because the differences along the two paths now matter. Due to such additional considerations the future cutoff levels depend not only on how many agents have adopted by then but also on when they (including the agent in question) adopted. This implies that the final network can be different depending on when the agent in question adopts. So, Lemma 4 no longer holds. Nevertheless, the effects of these complications become negligible as δ approaches 1 because then the discrepancy in argument from the case $\delta = 1$ either happens with a negligible probability since the cutoff levels differ only marginally, or has a negligible effect since it applies only to a finite number of periods before the same, terminal network is reached. Therefore, the basic insights behind Theorem 1 extends to large values of δ: There is a unique symmetric equilibrium that satisfies the stopping rule, and it is a cutoff equilibrium, however the cutoff levels now depend on the full adoption history (rather than only on the number of total adopters) up to then.

The basic picture stays the same when the stopping rule is lifted as well, although the details are more complex. In particular, multiple symmetric equilibria may exist in some environments but the differences are insignificant when the agents are patient, and they all converge to the unique symmetric equilibrium of Theorem 1 as δ tends to 1.[11] These results are formally stated in the next theorem and are proved in the Appendix.

Theorem 2: *Suppose that $\dot{u}_t(n)$, the derivative of $u_t(n)$ with respect to t, exists and $|\dot{u}_t(n)| > \theta$ for all $t \in [0, \bar{t}]$ and $n = 1, \cdots, N$, for some $\theta > 0$.*

(a) *There is $\delta^* < 1$ such that if $\delta \in (\delta^*, 1)$ there is a unique symmetric equilibrium of Γ that satisfies the stopping rule. This equilibrium is a cutoff equilibrium and converges to the equilibrium described in Theorem 1 as $\delta \to 1$.*

(b) *For any $\epsilon > 0$ and $\eta > 0$, there is $\delta_{\epsilon\eta} < 1$ such that if $\delta \in (\delta_{\epsilon\eta}, 1)$ then the followings hold in any symmetric equilibrium of Γ: For all $s = 1, \cdots, N$, (i) all agents will have adopted within $\kappa(\delta)$ periods from the state reaching s if their types are below $\tau_s - \epsilon$ but would never adopt while the state stays at s if their types exceed $\tau_s + \epsilon$, where τ_s is the equilibrium cutoff level of Theorem 1 and $\kappa(\delta)$ is the largest integer satisfying $\delta^{\kappa(\delta)} > 1 - \eta$, and (ii) for all $t \geq 0$ the equilibrium payoff of a t-type agent is within ϵ of that in the equilibrium of Theorem 1.*

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[11] In a previous version of this paper it has also been shown that the equilibrium of Theorem 1 is the unique symmetric equilibrium in a continuous-time model *a la* Simon and Stinchcombe [21] for all instantaneous discount factor in $(0, 1)$.

The proof of Theorem 2 follows the same core structure of induction as that in the proof of Theorem 1. For part (a), the key technical difference is that the entire range of possible adoption paths should be considered in comparing the payoffs from adopting and not adopting in the current period, rather than only those that lead to different final networks as was done in proving Theorem 1. However, for δ near 1, the possible paths that may ensue from either option differ only slightly from those when $\delta = 1$ and thus, so do both sides of the equation that determine the equilibrium cutoff type. Consequently, the solution at δ near 1 stays unique and differs only slightly by continuity, provided that the solution at $\delta = 1$ is nonsingular, which is guaranteed if $|\dot{u}_t(n)| > \theta$.

When the stopping rule is lifted, an additional source of complication is intertemporal coordnation within the same state. Indeed, it may be possible that different types would wait different length of time before adopting after a certain state has been reached, if they differ sufficiently in evaluating different potential adoption paths. This means that we have to allow for non-cutoff strategies. Nonetheless, in each induction step, given that any adoption would lead to a continuation equilibrium close to that in Theorem 1, any departure from the cutoff structure is a local phenomenon that occurs near the equilibrium cutoff level of Theorem 1, as shown in Appendix. Consequently, it is shown that optimal adoption decisions differ from those in Theorem 1 only locally both type-wise and time-wise for δ near 1, establishing part (b) of Theorem 2.

Up to now we have focused on symmetric equilibrium and characterized it fully. We note, however, that equilibria in asymmetric cutoff strategies may also exist in some environments (although they seem rather contrived given that all agents are ex ante identical). That is, different cutoff levels of two agents in a certain period may be mutual best responses because the agent with a higher (lower) cutoff level would, conditional on him being the sole adopter in that period, find that the other agent, whose posterior type distribution being more (less) favorable, is more (less) likely to adopt in the next period.[12]

Note, however, that each agent is optimizing relative to the strategies of all other agents and that for any two agents their respective sets of "all other agents" differ only with respect to each other. Since this difference gets insignificant as the number of agents, N, grows without bound,[13] so do the differences in the strategy profiles of "all other agents"

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[12] We provide a two-agent example: $u_t(n) = n - 1 - t$ is the utility functions for $n = 1, 2$, and F satisfies $F(0.2) = 1/6$, $F(0.4) = 3/8$ and $F(1) = 1/2$. Clearly, $\bar{t} = 1$ is the cutoff level when only one agent remains. Let t_1 and t_2 be the cutoff levels of agents 1 and 2, respectively, when $\delta = 1$ and neither of them adopted, i.e., in state $s = 2$. The equilibrium requires that agent 1 of t_1 type be indifferent between adopting and not, i.e., $F(1)u_{t_1}(2) + (1 - F(1))u_{t_1}(1) = F(t_2)u_{t_1}(2)$, or equivalently, $(\frac{1}{2} - F(t_2))(1 - t_1) = (1 - \frac{1}{2})t_1$. The analogous equilibrium condition for agent 2 is $(\frac{1}{2} - F(t_1))(1 - t_2) = (1 - \frac{1}{2})t_2$. One can easily verify from $F(0.2) = 1/6$ and $F(0.4) = 3/8$ that these two conditions are satisfied when $t_1 = 0.2$ and $t_2 = 0.4$ and when $t_1 = 0.4$ and $t_2 = 0.2$, hence asymmetric cutoff equilibria exist.

[13] This is so even if the adoption process went a long way and only a small number of agents remain, because "all other remaining agents" can be any subset of the initial set of agents with the right cardinality. Here, we implicitly assume anonymity in the sense that each agent cannot tell other agents apart except

relative to which each agent optimizes. Consequently, all asymmetric equilibria converge to the unique symmetric equilibrium, as elaborated in Theorem 4 of the next Section.

4. Properties of the Equilibrium and Asymptotic Efficiency

Due to the agents' types being private information, the final network to be formed by the equilibrium adoption process is stochastic, i.e., it depends on the realized type-profile of the agents. Although every adoption is optimal when the decision is made, the actual network to be formed may not be ex post efficient as explained in the Introduction: some adopters may end up in a final network too small for them to benefit from (this is a risk they find worth taking when they adopt before the network grows sufficiently big, lest the adoption process runs out of stream prematurely to preclude possible surges later in the adoption process), and/or there may be some non-adopters who would adopt if they knew one another's types. This ex post inefficiency arises as a consequence of the adoption decisions being irreversible and thus, can be eliminated if the agents are allowed to reverse their adoption decisions at no cost as elaborated in the next section, so long as the agents are sufficiently patient.

Furthermore, even when adoptions are irreversible the ex post inefficiency vanishes asymptotically as the population size grows large, because the equilibrium adoption path becomes arbitrarily certain due to the law of large numbers. Since it has been established in the previous section that all symmetric equilibria converge to the unique symmetric equilibrium identified in Theorem 1 as δ tends to 1, we formalize the asymptotic efficiency result for the case $\delta = 1$ in Theorem 4 below, which follows from the comparative statics result, to be stated in Theorem 3, on the equilibrium cutoff levels as N changes.

A change of notation proves useful. We suppressed the total population size, N, in the notation τ_s, the equilibrium cutoff level when there are s remaining agents. To conduct comparative statics on N, however, we need to make N explicit: We use $\tau_{n|N}$ to denote the unique equilibrium cutoff level under $\delta = 1$ after n agents have adopted when the population size is N. Note that the subscript n in $\tau_{n|N}$ refers to the number of adopters, rather than non-adopters, after a certain history. This proves useful because the former stays constant for the same history regardless of N, while the latter varies.

Theorem 3: *Given F and $u_t(\cdot)$, $\tau_{n|N} < \tau_{n|N+1}$ for all $N > 1$ and $n = 0, 1, \cdots, N-1$.*

Proof. See Appendix.

Corollary: *When $\delta = 1$, the distribution of final network size improves in the first-order stochastic sense as the number of agents increases.*

Theorem 3 states that the equilibrium cutoff level after any possible history, including the null one, increases as the population size increases. Suppose hypothetically that the

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by their past adoption decisions.

cutoff levels remained the same when the population, denoted by $\mathcal{I}$, grew from a size N to $N+1$. Consider a sub-population $\mathcal{I}' \subset \mathcal{I}$ consisting of N agents. For any feasible type profile of $\mathcal{I}$, the agents in $\mathcal{I}'$ who would adopt in the equilibrium were $\mathcal{I}'$ the entire population would also adopt in the game with the population $\mathcal{I}$. Since an additional agent in $\mathcal{I}$ and the increased equilibrium cutoff levels increase the network size to be formed eventually, the Corollary ensues.

If the population size N continues to increase, so does the cutoff levels $\tau_{n|N}$ for each n by Theorem 3. For any fixed n, if $\tau_{n|N}$ increases without bound, then an arbitrarily high fraction of the whole population would have entered by the time the network size reaches $n+1$ if N is sufficiently large, because $F(t) \to 1$ as $t \to \infty$.

Consider the alternative case that $\tau_{n|N}$ converges, say to $\tau_{n|\infty} < \infty$, as $N \to \infty$. If, in addition, $\tau_{n+1|N}$ converges to $\tau_{n+1|\infty} > \tau_{n|\infty}$ such that $F(\tau_{n+1|\infty}) > F(\tau_{n|\infty})$, then the expected number of adopters in the period following the network size has reached n would exceed any finite number with virtual certainty as $N \to \infty$ by the law of large numbers. This would imply that if N is sufficiently large, virtually any remaining agent who might ever adopt would adopt in that period, which would include agents of types between $\tau_{n|N}$ and $\tau_{n+1|N}$, contradicting $F(\tau_{n+1|\infty}) > F(\tau_{n|\infty})$. Hence, we deduce that if $\tau_{n|N}$ converges to a finite limit $\tau_{n|\infty}$ as $N \to \infty$, then $\tau_{n+1|N}$ converges to a limit, $\tau_{n+1|\infty}$, such that $F(\tau_{n+1|\infty}) = F(\tau_{n|\infty})$. Since this applies to all n, it follows that as the population gets large, virtually all agents who might ever adopt will have adopted by the time the network reaches size n. For $n = 1$, in particular, this means that they all will adopt in the initial period, establishing the ex post efficiency asymptotically as the population size increases. We formalize this result in Theorem 4 below. It is worth mentioning that this result is obtained for asymmetric equilibria as well: Part (b) implies that the fraction of agents who would not adopt in the initial period when they have any chance of adopting at all, vanishes asymptotically as $N \to \infty$.

Theorem 4: *Fix F and $u_t(\cdot)$. For $N > 1$, let $\bar{t}_N$ be the unique solution to $u_t(N) = 0$.*

$$(a) \qquad \frac{F(\tau_{0|N})}{F(\bar{t}_N)} \to 1 \quad as \quad N \to \infty.$$

(b) Given any sequence $\{(a_N^i)_{1 \le i \le N}\}_{N=2}^{\infty}$ of asymmetric equilibria, define $\tau_{0|N}^i := \inf\{\, t \mid a_N^i(t|h_1) < 1\}$ and $\tau_{0|N}^{\chi} := \inf\{t \mid \#\{i|\tau_{0|N}^i > t\}/N < \chi\}$ for $\chi < 1$. Then, $\frac{F(\tau_{0|N}^{\chi})}{F(\bar{t}_N)} \to 1$ as $N \to \infty$, for any $\chi < 1$.

We close this section with a second comparative statics result that compares two populations of the same size, in one of which the type is more favorably distributed than in the other: F is "more favorable" than $\tilde{F}$ if an agent is more likely to be at least as enthusiastic as any given type according to F than according to $\tilde{F}$, i.e., $\tilde{F}$ first-order stochastically dominates F. The next result confirms the intuition that the more enthusiatic the agents are the more aggressively they will adopt, and consequently, a larger network forms on average.

Theorem 5: *Given N, let τ_s and $\tilde{\tau}_s$ denote the cutoff levels of the unique symmetric equilibria for state s when $\delta = 1$ and the type distributions are F and $\tilde{F}$, respectively, where F is more favorable than $\tilde{F}$, i.e., $F(t) > \tilde{F}(t)$ for all $t > 0$. Then, $F(\tau_s) > \tilde{F}(\tilde{\tau}_s)$ for all s.*

Proof. See Appendix.

5. The Case of Reversible Adoption

As explained above, the unique symmetric equilibrium is not generally ex post efficient. Clearly, one reason that the final network can contain an unhappy member is because he is not allowed to leave once he joins. In this section we explore the extent to which the ex post efficiency can be improved when the agents are allowed to leave the network, reversing their decisions. The findings are two-fold: The ex post efficiency can indeed be achieved in one set of equilibria; But, other equilibria also exist, including those in which the complete coordination failure resurfaces, i.e., the beliefs that no one will ever adopt are self-fulfilling. We demonstrate this result below for the case that $\delta = 1$.

It is straightforward to see that the no-adoption equilibrium is sustained by the following beliefs: Every agent believes that no agent will adopt in any period, and in cases that any agent is in the network in any period (along the off-equilibrium paths) all other agents believe that the adopters will reverse their decisions and exit in the following period.[14] Note that such beliefs are not possible when the adoptions are irreversible, hence neither is the no-adoption equilibrium.

As for the ex post efficient equilibrium, we construct one as follows. In the first period, every agent employs a cutoff strategy at $\bar{t}$, i.e., adopts precisely when he is happy to be a member of the full network. If either all adopt or none does in the first period, there is no change in the network in any subsequent period. Otherwise, i.e., if some, say n, but not all agents adopted in the first period, then in the second period those who would be happy to be a member in case the current network prevails, stays in the network and all others leave (i.e., the cutoff level is the type whose payoff from a network of size n is 0). If all or none stayed in the network in the second period, no further change in the network takes place in any subsequent period. Otherwise, i.e., if some but not all agents stayed, then in the third period those who would be happy to be a member of the then-current network, stays in the network and all others leave. The change in the network continues in the same

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[14] Even when some agent did not exit in the last period (contrary to the off-equilibrium belief), other agents continue to believe that he will exit in the next period.

manner in subsequent periods until either the network did not change from the previous period or everyone leaves the network. It is a straightforward exercise (hence is omitted) to verify that this constitutes an equilibrium that is ex post efficient.

We described two extreme equilibria when the adoption decisions are reversible, one with no adoption at all due to ill-coordinated beliefs, and the other in which the agents achieve ex post efficiency, by going for the best possible network initially, then “downsizing” subsequently as needed, aiming at the next best possible network that the adoption history indicates. There are still other equilibria between these polar cases. For example, equilibria exist in which the agents initially go for a medium-size network, then go for a larger network if the result is promising, or start downsizing otherwise.

6. Concluding remarks

New products and services that have network externalities are often adopted by at least some of the potential users of such products and services. A satisfactory model of the adoption process should be able to account for the size of the group that chooses to enter a network. A static model of network formation cannot do this because the existence of complementarities implies that there are a multiplicity of equilibria. However, it is natural to think of the formation of a network as a dynamic process in which each agent can observe at each moment in time how many people have already entered the network and can use this information to update his/her beliefs with respect to the expected number of additional agents who will eventually join the network. When the time an agent enters is endogenously determined, the entry decision is irreversible and the discount factor is sufficiently close to one there is essentially a unique symmetric equilibrium. This equilibrium is inconsistent with the belief that no one will enter. Furthermore, if the number of agents is large, all asymmetric equilibria (if they exist) are approximated by the unique, symmetric equilibrium. Therefore, modeling the entry process as a dynamic game of incomplete information is not only more realistic, but, as our analysis of a simple dynamic market entry game shows, can eliminate the coordination problem associated with the static market entry game.

Because the model has a unique equilibrium it is possible to derive testable comparative static predictions. Two testable implications are: (1) The more enthusiastic is the population about adopting the network (in the sense of first-order stochastic dominance), the greater is the expected number of people who will enter in equilibrium; (2) The larger the number of potential entrants the smaller the number of previous entrants an agent of any type must observe prior to entering.[15] The latter has a further consequence that, as

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[15] Possible data to test this prediction, for instance, would be telephone adoption in cities of different sizes prior to long-distance service (we thank an anonymous referee for this suggestion). For the first prediction, data on membership of social networking websites for different age groups might do.

the population size increases, virtually all agents who might ever enter will indeed enter in the initial period, thereby establishing asymptotic ex post efficiency of the unique outcome.

The assumption of the irreversibility of the entry decision is critical for our game to have a unique symmetric equilibrium. If entry decisions were reversible the impact of an agent's entry decision on the subsequent entry decisions of other agents would be diminished by the possibility that an earlier entrant might subsequently leave. This possibility permits the existence of a multiplicity of equilibria when actions are reversible, including a no-entry equilibrium as well as an ex post efficient equilibrium.

In our model there is only one possible network. However, we sometimes observe the co-existence of competing networks, in each of which the benefits to its members are increasing in the size of its membership. Suppose that any agent who enters a particular network is committed to that network for the remainder of that agent's "life", each agent's life is finite, and there are overlapping generations, in which each newly born agent must choose a strategy to determine which network to join and when to join that network, if ever. Such a model would preserve the irreversibility of an entry decision of each agent, but would not insure any agent who enters a network that the size of membership in that network cannot decrease at some later date. Is this sufficient to generate equilibria in which two networks co-exist in a steady state or is the co-existence of competing networks necessarily only a transitory phenomenon? We intend to address these questions in the future.

Appendix

Proof of Theorem 2. In this proof we use some equilibrium features of $\Gamma_{\tilde{s}}(g)$, the continuation game starting from an arbitrary period of state $\tilde{s}$ with a posterior density function g when there is no discounting ($\delta = 1$). As shown in Section 3 (see property [A]), in the unique symmetric equilibrium of the continuation game $\Gamma_{\tilde{s}}(g)$ the (remaining) agents employ the cutoff strategies at the cutoff levels, $\tau_s(g)$, that depend only on the state s (given g) and $0 \leq \tau_{\tilde{s}}(g) < \tau_{\tilde{s}-1}(g) < \cdots < \tau_1(g) = \bar{t}$. We refer to this equilibrium as the "focal" equilibrium of the continuation game $\Gamma_{\tilde{s}}(g)$. We use $v^*_{\tilde{s}}(t|g)$ to denote the expected payoff of a remaining agent of t-type in the focal equilibrium of $\Gamma_{\tilde{s}}(g)$.

Observe that the process of calculating $\tau_s(g)$ in Section 3 can be applied for all $s = 1, \cdots, N$, although only $\tau_s(g)$'s with $s \leq \tilde{s}$ are relevant in the continuation game $\Gamma_{\tilde{s}}(g)$. Since the level of $\tau_s(g)$ is independent of $\tilde{s}$ for all $s = 1, \cdots, N$, we use $\tau_s(g)$ without reference to $\tilde{s}$.

Lemma A1: *There is* $\lambda > 0$ *such that* $\tau_{s+1}(g) < \tau_s(g) - \lambda$ *for all* g *that satisfy (3) and* $s = 1, \cdots, N-1$.

Proof. To prove by contradiction, suppose to the contrary that there exists a sequence $\{g^\ell\}_{\ell=1}^\infty$ such that $\tau_{s'}(g^\ell) - \tau_{s'+1}(g^\ell) \to 0$ as $\ell \to \infty$ for some $s' = 1, \cdots, N-1$. Recall that $\tau_{s'}(g^\ell)$ and $\tau_{s'+1}(g^\ell)$ are the solutions to the equation (7) when $g|_{t \geq \hat{t}} = g^\ell|_{t \geq \hat{t}}$ and $\tilde{r} = s'$ and $\tilde{r} = s' + 1$, respectively, where $\text{Prob}(j|g^\ell|_{t \geq \hat{t}})$'s represent the distribution of final adopters when there are $s' - 1$ remaining agents and s' remaining agents, respectively, whose types are distributed according to $g^\ell|_{t \geq \hat{t}}$. Since the remaining agents, in either case, adopt according to the same cutoff levels, namely, $\tau_s(g^\ell)$ for $s \leq s'$, the distribution of final adopters when there are s' remaining agents is worse, in the sense of first-order stochastic dominance, than that when there are $s' - 1$ remaining agents (because one more agent has adopted already in the latter case). Since $u_t(n)$ strictly increases in n for all $t \geq 0$, for $\tau_{s'}(g^\ell) - \tau_{s'+1}(g^\ell) \to 0$ to hold as $\ell \to \infty$, therefore, the distribution of final adopters in the two cases should converge as $\ell \to \infty$: in particular, the probabilities of the event that the final network size is at least $N - s' + 1$ in the two cases should converge. Note that this probability is 1 if only $s' - 1$ agents remain whilst it is at most $1 - (\int_{\bar{t}}^\infty g^\ell|_{t \geq \hat{t}} dt)^{s'} \leq 1 - (\int_{\bar{t}}^\infty f dt)^{s'} < 1$ by (3) if s' agents remain, a contradiction. □

We use $\Gamma_{\tilde{s}}(g|\delta)$ to denote the continuation game $\Gamma_{\tilde{s}}(g)$ when the discount rate is $\delta \leq 1$. We continue to impose the condition (3) on g for the same reason as explained earlier.

(a) In the proof of part (a) presented below, we take it granted that we only consider equilibria that satisfy the stopping rule (i.e., without mentioning it). We characterize the properties satisfied by any symmetric equilibrium $(a^i)_{i \in \mathcal{I}}$ of Γ for sufficiently large $\delta < 1$ by an induction argument. The part (a) of Theorem 2 will be proved as a result.

STEP A1: It is trivial that if only one agent remains in some period k, then for all $\delta < 1$ this last agent will adopt precisely when his type does not exceed $\bar{t}$ defined in (2). That is, the equilibrium strategy in the continuation game $\Gamma_1(g|\delta)$ is a cutoff strategy at $\tau_1(g|\delta) \equiv \bar{t}$.

STEP A2: Consider a continuation game $\Gamma_2(g|\delta)$ starting from period k of the original game. Consider one remaining agent, say i, of type $t_i \leq \tau_1(g|\delta) = \bar{t}$. If the other remaining agent, say j, were to adopt in this period (which happens with probability, say π), agent i would get a stage utility of $u_{t_i}(N)$ forever by adopting in this period; if agent i waited in this period he would adopt in the next period (because $t_i \leq \bar{t}$), hence again get a stage utility of $u_{t_i}(N)$ forever but from next period onwards.

Next consider the contingency that agent j were to not adopt in this period, which happens with probability $1 - \pi$. In this contingency, agent i would get a utility $u_\phi = 0$ by not adopting in this period; if agent i adopted in this period, he would get $u_{t_i}(N-1)$ this period, and from next period on he would get $u_{t_i}(N)$ in case agent j joins next period (which happens with probability, say p, conditional on j does not join in this period) and

$u_{t_i}(N-1)$ otherwise. Note that the agent j's response in the next period is independent of t_i. Combining the two contingencies, the benefit of adopting in this period as opposed to waiting is

$$\pi(1-\delta)u_{t_i}(N) + (1-\pi)\Big[u_{t_i}(N-1) + p\delta(u_{t_i}(N) - u_{t_i}(N-1))\Big]$$

which is strictly decreasing in t_i regardless of π and p, with a negative value at $t_i = \bar{t}$. Hence, agent i (and j by symmetry) should employ a cutoff strategy at a level, say $\hat{t} < \bar{t}$.

Since $\pi = \int_0^{\hat{t}} g(t)dt$ and $p = \int_{\hat{t}}^{\tau_1} g(t)dt / \int_{\hat{t}}^{\infty} g(t)dt$, the cutoff level $\hat{t}$ satisfies

$$\begin{aligned}
&(1-\delta)u_{\hat{t}}(N)\int_0^{\hat{t}} g(t)dt \\
&\qquad + \big(1 - \int_0^{\hat{t}} g(t)dt\big)\Big[u_{\hat{t}}(N-1) + \frac{\delta\int_{\hat{t}}^{\tau_1} g(t)dt}{\int_{\hat{t}}^{\infty} g(t)dt}(u_{\hat{t}}(N) - u_{\hat{t}}(N-1))\Big] \;=\; 0 \\
\iff\quad & \psi(\hat{t}|g,\delta) := (1-\delta)u_{\hat{t}}(N)\int_0^{\hat{t}} g(t)dt\int_{\hat{t}}^{\infty} g(t)dt + \Big(1 - \int_0^{\hat{t}} g(t)dt\Big)\times \\
&\qquad \Big[u_{\hat{t}}(N-1)\Big(\int_{\hat{t}}^{\infty} g(t)dt - \delta\int_{\hat{t}}^{\tau_1} g(t)dt\Big) + \delta u_{\hat{t}}(N)\int_{\hat{t}}^{\tau_1} g(t)dt\Big] \;=\; 0.
\end{aligned} \tag{8}$$

Note that as $\delta \to 1$, i) the first term of $\psi(\hat{t}|g,\delta)$ becomes negligible, and ii) the second term is strictly decreasing in $\hat{t}$ (with the derivative bounded away from 0), clearly from a positive value when $\hat{t} = 0$ to a negative value when $\hat{t} = \tau_1 (= \bar{t})$. Hence, for δ sufficiently close to 1 there exists a unique value of $\hat{t}$ that solves (8), which is the equilibrium cutoff level in the first period of the continuation game $\Gamma_2(g|\delta)$, denoted by $\tau_2(g|\delta)$. Furthermore, since the first derivative of $\psi(\hat{t}|g,\delta)$ when $\delta = 1$ is bounded away from 0 by a number independent of g (because this derivative is bounded above by $\max_t \dot{u}_t(N-1) \leq -\theta < 0$), by continuity, for any $\epsilon > 0$ there exists $\delta_\epsilon < 1$ (independent of g) such that if $\delta > \delta_\epsilon$ then the first derivative of $\psi(\hat{t}|g,\delta)$ is bounded away from 0 for all $t \leq \bar{t}$ and consequently, $\tau_2(g|\delta)$ uniquely exists and $|\tau_2(g|\delta) - \tau_2(g)| < \epsilon$ where $\tau_2(g)$ is, as defined earlier, the solution to (4), or equivalently, the solution to (8) when $\delta = 1$.

Furthermore, consider g's in the form of $f|_{t \geq \underline{t}}$ for $\underline{t} < \tau_2(f)$ and treat $\psi(\hat{t}|g,\delta)$ as a function of $\hat{t}$ and $\underline{t}$, denoted by $\psi(\hat{t},\underline{t}|\delta)$. Let $\hat{t}^*(\underline{t})$ be the solution to (8) when $g = f|_{t \geq \underline{t}}$, i.e., $\hat{t}^*(\underline{t}) = \tau_2(f|_{t \geq \underline{t}}|\delta)$, which uniquely exists for $\delta < 1$ sufficiently large as shown above. Since $\psi(\hat{t},\underline{t}|\delta)$ is continuously differentiable in both arguments and the first partial derivative $\psi_1(\hat{t}^*(\underline{t}),\underline{t}|\delta)$, which coincides with the first derivative of $\psi(\hat{t}|g,\delta)$, is bounded away from 0 for $\delta > \delta_\epsilon$ as shown above, by the Implicit Function Theorem, $d\hat{t}^*(\underline{t})/d\underline{t} = -\psi_2(\hat{t}^*(\underline{t}),\underline{t}|\delta)/\psi_1(\hat{t}^*(\underline{t}),\underline{t}|\delta)$ where ψ_i denotes the i-th partial derivative of $\psi(\hat{t},\underline{t}|\delta)$. Since

$$\psi_2(\hat{t},\underline{t}|\delta) = f(\underline{t})\Big[u_{\hat{t}}(N-1)\Big(\int_{\hat{t}}^{\infty} g(t)dt - \delta\int_{\hat{t}}^{\tau_1} g(t)dt\Big) + \delta u_{\hat{t}}(N)\int_{\hat{t}}^{\tau_1} g(t)dt - (1-\delta)u_{\hat{t}}(N)\int_{\hat{t}}^{\infty} g(t)dt\Big]$$

we deduce from $\psi(\hat{t}^*(\underline{t}), \underline{t}|\delta) = 0$ that the above expression evaluated at $\hat{t} = \hat{t}^*(\underline{t})$ vanishes as $\delta \to 1$, i.e., $\psi_2(\hat{t}^*(\underline{t}), \underline{t}|\delta) \to 0$ uniformly as $\delta \to 1$, and consequently, $d\hat{t}^*(\underline{t})/d\underline{t} \to 0$ uniformly as $\delta \to 1$. Together with the STEP A1, we have proved the property [A'] stated below for $\tilde{s} = 2$.

STEP A3: Given a cutoff equilibrium of a continuation game $\Gamma_{\tilde{s}}(g|\delta)$, let $\tau_s(g, h|\delta)$ denote the equilibrium cutoff level in a period of state $s(\leq \tilde{s})$ after a history h.

[A'] For any $\epsilon > 0$, there is $\delta_\epsilon(\tilde{s}) < 1$ such that the followings hold if $\delta \in (\delta_\epsilon(\tilde{s}), 1)$ for any $s \leq \tilde{s}$ and any history h whose state is s: (i) Any symmetric equilibrium of any continuation game $\Gamma_{\tilde{s}}(g|\delta)$ is a cutoff equilibrium, and $|\tau_s(g, h|\delta) - \tau_s(g)| < \epsilon$; (ii) Any continuation game $\Gamma_{\tilde{s}}(f|_{t\geq \underline{t}}|\delta)$ has a unique symmetric equilibrium and $d\tau_s(f|_{t\geq \underline{t}}, h|\delta)/d\underline{t} < \epsilon$ for all $\underline{t} < \tau_{\tilde{s}}(f) - \lambda/2$ where λ is the constant identified in Lemma A1.

We now make an induction hypothesis that the property [A'] holds for all $\tilde{s} < \tilde{r}$ where $\tilde{r} = 3, \cdots, N$, along an equilibrium. Then, we establish that under this hypothesis the property [A'] holds for $\tilde{s} = \tilde{r}$ as well.

To do this, fix $\epsilon > 0$ and consider an arbitrary symmetric equilibrium of a continuation game $\Gamma_{\tilde{r}}(g|\delta)$. Let g' denote the equilibrium density after the first period of this continuation game. By the induction hypothesis,

$$\forall \epsilon' > 0, \quad |\tau_s(g', h|\delta) - \tau_s(g')| < \epsilon' \quad \forall s \leq \tilde{r} - 1 \quad \text{if} \quad \delta \in (\delta_{\epsilon'}(\tilde{r} - 1), 1). \tag{9}$$

Consider a remaining agent i of type $t_i < \tau_{\tilde{r}-1}(g'|\delta)$ in the first period of the continuation game $\Gamma_{\tilde{r}}(g|\delta)$. The expected utility of this agent from adopting in this period is

$$\begin{aligned}
&\sum_{\ell=1}^{\tilde{r}-1} \Big(\delta^{\ell-1}(1-\delta) \sum_{j=0}^{\tilde{r}-2} a_{\ell,j} u_{t_i}(N - r + j + 1) \Big) \\
&+ \pi \sum_{\substack{1 \leq \ell \leq \tilde{r} \\ 1 \leq j \leq \tilde{r}-1}} p^+_{\ell,j} \delta^{\ell-1} u_{t_i}(N - r + j + 1) + (1-\pi) \sum_{\substack{1 \leq \ell \leq \tilde{r} \\ 0 \leq j \leq \tilde{r}-1}} p^0_{\ell,j} \delta^{\ell-1} u_{t_i}(N - r + j + 1)
\end{aligned} \tag{10}$$

where $a_{\ell,j}$ is the probability, conditional on the agent i adopting in the current period, that the network size in period ℓ is $N - r + j + 1$ and further adoption ensues; π is the probability that some other agent adopts in period 1 of the continuation game $\Gamma_{\tilde{r}}(g|\delta)$; $p^+_{\ell,j}$ ($p^0_{\ell,j}$) is the probability, conditional on the agent i adopting in period 1 and some (no) other agent adopting in period 1, that the network size reaches $N - r + j + 1$ in period ℓ and no further adoption ensues. The expected utility of this agent from waiting in this period is

$$\pi \sum_{\ell=2}^{\tilde{r}-1} \Big(\delta^{\ell-1}(1-\delta) \sum_{j=0}^{\tilde{r}-2} b_{\ell,j} u_{t_i}(N - r + j + 1) \Big) + \pi \sum_{\substack{1 \leq \ell \leq \tilde{r} \\ 1 \leq j \leq \tilde{r}-1}} q_{\ell,j} \delta^{\ell-1} u_{t_i}(N - r + j + 1) \tag{11}$$

where $b_{\ell,j}$ is the probability, conditional on the agent i not adopting in the current period and at least one other agent does, that the network size in period ℓ is $N-r+j+1$ and further adoption ensues; and $q_{\ell,j}$ is the probability, conditional on the same contingency, that the network size reaches $N-r+j+1$ in period ℓ and no further adoption ensues. Therefore, the expected benefit of adopting in this period as opposed to waiting is

$$\sum_{\ell=1}^{\tilde{r}-1}\Big(\delta^{\ell-1}(1-\delta)\sum_{j=0}^{\tilde{r}-2}(a_{\ell,j}-\pi b_{\ell,j})u_{t_i}(N-r+j+1)\Big) + \sum_{\substack{1\le\ell\le\tilde{r}\\1\le j\le\tilde{r}-1}}\pi(p^{+}_{\ell,j}-q_{\ell,j})\delta^{\ell-1}u_{t_i}(N-r+j+1)+(1-\pi)\sum_{\substack{1\le\ell\le\tilde{r}\\0\le j\le\tilde{r}-1}}p^{0}_{\ell,j}\delta^{\ell-1}u_{t_i}(N-r+j+1). \tag{12}$$

Observe that $\pi \le 1-(1-G(\bar{t}))^{\tilde{r}-1}<1$ by (3), and that $a_{\ell,j}$, $b_{\ell,j}$, $p^{+}_{\ell,j}$, $p^{0}_{\ell,j}$, and $q_{\ell,j}$ are all polynomials in terms of $\int_0^{\tau_s(g',h|\delta)} g'(t)dt$ for $s<\tilde{r}$ and relevant h.

First, consider the contingency that at least one other agent adopts in period 1 of $\Gamma_{\tilde{r}}(g|\delta)$. In the hypothetical case that $\tau_s(g',h|\delta)=\tau_s(g')$ for all $s<\tilde{r}$ and h whose state is s, if the agent i did not adopt in this period, by adopting in the next period he can ensure the same final network size as the one that would have resulted if he adopted in the first period, by the same argument as the proof of Lemma 4. By (9), therefore, the following holds for δ sufficiently close to 1: If the agent i did not adopt in period 1 of $\Gamma_{\tilde{r}}(g|\delta)$, by adopting in the next period he can ensure with arbitrarily large a probability the same final network size as the one that would have resulted if he adopted in period 1. This means that

$$(p^{+}_{\ell,j}-q_{\ell,j})\to 0 \quad \text{for all} \quad \ell,j\ge 1 \quad \text{as} \quad \delta\to 1. \tag{13}$$

Thus, the derivative of (12) with respect to t_i uniformly converges to

$$(1-\pi)\sum_{\substack{1\le\ell\le\tilde{r}\\0\le j\le\tilde{r}-1}}p^{0}_{\ell,j}\delta^{\ell-1}\dot{u}_{t_i}(N-r+j+1)\le-(1-\pi)\theta<0 \quad \text{as} \quad \delta\to 1 \tag{14}$$

because $\sum_{\substack{1\le\ell\le\tilde{r}\\0\le j\le\tilde{r}-1}}p^{0}_{\ell,j}=1$. Since the agent i would adopt in the first period of the continuation game $\Gamma_{\tilde{r}}(g|\delta)$ if and only if the value of (12) is positive, therefore, the equilibrium strategy in that period is a cutoff strategy for sufficiently large δ.

The equilibrium cutoff level, $\tau_{\tilde{r}}(g|\delta)$, is the value of $\hat{t}$ that solves $\Psi(\hat{t}|g,\delta)=0$ where

$$\Psi(\hat{t}|g,\delta):=\sum_{\ell=1}^{\tilde{r}-1}\Big(\delta^{\ell-1}(1-\delta)\sum_{j=0}^{\tilde{r}-2}(a_{\ell,j}-\pi b_{\ell,j})u_{\hat{t}}(N-r+j+1)\Big) + \sum_{\substack{1\le\ell\le\tilde{r}\\1\le j\le\tilde{r}-1}}\pi(p^{+}_{\ell,j}-q_{\ell,j})\delta^{\ell-1}u_{\hat{t}}(N-r+j+1)+(1-\pi)\sum_{\substack{1\le\ell\le\tilde{r}\\0\le j\le\tilde{r}-1}}p^{0}_{\ell,j}\delta^{\ell-1}u_{\hat{t}}(N-r+j+1) \tag{15}$$

where $\pi = 1-(1-\int_0^{\hat{t}} g dt)^{\tilde{r}-1}$ and $a_{\ell,j}$, $b_{\ell,j}$, $p^{+}_{\ell,j}$, $p^{0}_{\ell,j}$, and $q_{\ell,j}$ are calculated for $g' = g|_{t \geq \hat{t}}$. If $\delta = 1$, the first two of the three sums of Ψ are identically 0 and thus, the derivative of Ψ with respect to $\hat{t}$ is bounded away from 0 by (14) because $p^{0}_{\ell,j}$'s, being functions of $\tau_s(g|_{t \geq \hat{t}})$ for $s < \tilde{r}$, are fixed independently of $\hat{t} < \tau_{\tilde{r}-1}(g)$ when $\delta = 1$ as per [A] (ii) that has been proved in Section 3. By continuity, therefore, as $\delta \to 1$ the solution to $\Psi = 0$ converges to that when $\delta = 1$, that is, $\tau_{\tilde{r}}(g|\delta) \to \tau_{\tilde{r}}(g)$ as $\delta \to 1$. Together with the induction hypothesis that [A'] holds for all $r < \tilde{r}$, we have proved property (i) of [A'] for $r = \tilde{r}$.

Consider the case that $g = f|_{t \geq \underline{t}}$ for some $\underline{t} < \tau_{\tilde{r}}(f) - \lambda/2$. Since $g' = f|_{t \geq \hat{t}}$ for $\hat{t} > \underline{t}$, by the induction hypothesis (ii) of [A'],

$$\forall s < \tilde{r}, \quad \frac{\partial \tau_s(g', h|\delta)}{\partial \hat{t}} \to 0 \text{ uniformly on } \ \hat{t} \in (\underline{t}, \tau_{\tilde{r}-1}(g'|\delta) - \lambda/2) \ \text{ as } \ \delta \to 1. \tag{16}$$

Applying the Chain Rule and (13), therefore, the derivative of the first two sums of (15) with respect to $\hat{t}$ vanishes as $\delta \to 1$ and, consequently, by (14) and continuity, Ψ strictly decreases in $\hat{t} \in (\underline{t}, \tau_{\tilde{r}-1}(g'|\delta) - \lambda/2)$ for δ sufficiently large, proving the uniqueness of $\tau_{\tilde{r}}(f|_{t \geq \underline{t}}|\delta)$.

Finally, for g's in the form of $f|_{t \geq \underline{t}}$, treat $\Psi(\hat{t}|g, \delta)$ as a function of $\hat{t}$ and $\underline{t}$, denoted by $\Psi(\hat{t}, \underline{t}|\delta)$. By applying the Chain Rule, it is straightforward (albeit lengthy) to verify that $\Psi(\hat{t}, \underline{t}|\delta)$ is continuously differentiable in both arguments. By (14) and (16), the first partial of $\Psi(\hat{t}, \underline{t}|\delta)$ is bounded away from 0 for sufficiently large $\delta < 1$. By the Implicit Function Theorem, therefore, $d\hat{t}^*(\underline{t})/d\underline{t} = -\Psi_2(\hat{t}^*(\underline{t}), \underline{t}|\delta)/\Psi_1(\hat{t}^*(\underline{t}), \underline{t}|\delta)$ where $\hat{t}^*(\underline{t})$ denotes the solution to $\Psi(\hat{t}|g, \delta) = 0$ when $g = f|_{t \geq \underline{t}}$ (which we have shown above to exist uniquely) and Ψ_i is the i-th partial derivative of $\Psi(\hat{t}, \underline{t}|\delta)$. Note that $a_{\ell,j}$, $b_{\ell,j}$, $p^{+}_{\ell,j}$, $p^{0}_{\ell,j}$, and $q_{\ell,j}$ are functions of $g' = f|_{t \geq \hat{t}}$, in particular, independent of $\underline{t}$ so long as $\underline{t} < \hat{t}$. Thus, for $\underline{t} < \hat{t}$,

$$\begin{aligned}\Psi_2(\hat{t}, \underline{t}|\delta) = \frac{\partial \pi}{\partial \underline{t}} \Bigg[& -\sum_{\ell=1}^{\tilde{r}-1} \Big(\delta^{\ell-1}(1-\delta) \sum_{j=0}^{\tilde{r}-2} b_{\ell,j} u_{\hat{t}}(N-r+j+1) \Big) + \\ & \sum_{\substack{1 \leq \ell \leq \tilde{r} \\ 1 \leq j \leq \tilde{r}-1}} (p^{+}_{\ell,j} - q_{\ell,j}) \delta^{\ell-1} u_{\hat{t}}(N-r+j+1) - \sum_{\substack{1 \leq \ell \leq \tilde{r} \\ 0 \leq j \leq \tilde{r}-1}} p^{0}_{\ell,j} \delta^{\ell-1} u_{\hat{t}}(N-r+j+1) \Bigg]\end{aligned}$$

where $\partial \pi / \partial \underline{t} = -f(\underline{t})(\tilde{r}-1)(1-\int_0^{\hat{t}} f|_{t \geq \underline{t}} dt)^{\tilde{r}-2}$. In conjunction with (13) and $\Psi(\hat{t}^*(\underline{t}), \underline{t}|\delta) = 0$, therefore, we deduce that $\Psi_2(\hat{t}^*(\underline{t}), \underline{t}|\delta) \to 0$ uniformly as $\delta \to 1$, and thus, $d\hat{t}^*(\underline{t})/d\underline{t} \to 0$ uniformly as $\delta \to 1$. This proves that the property (ii) of [A'] holds for $s = \tilde{r}$; Given this, together with the induction hypothesis, the Chain Rule implies that it holds for $s < \tilde{r}$ as well.

We have establishes the induction argument that if the property [A'] holds for all $\tilde{s} < \tilde{r} = 3, \cdots, N$, then [A'] holds for $\tilde{s} = \tilde{r}$ as well. Applying this result repeatedly, we conclude that [A'] holds for $\tilde{s} = N$, i.e., at the beginning of period 1, thereby establishing that there is a threshold $\delta^* < 1$ such that if $\delta \in (\delta^*, 1)$ then there is a unique symmetric equilibrium and this equilibrium converges to the unique equilibrium characterized in Theorem 1 as $\delta \to 1$. This completes the proof of part (a) of Theorem 2.

(b) We now consider any symmetric equilibrium, i.e., without subject to the stopping rule. In particular, we characterize any such equilibrium of the continuation games $\Gamma_{\tilde{s}}(g|\delta)$ recursively on $\tilde{s} = 1, \cdots, N$, which will eventually complete the proof.

Consider the following property where $\tilde{s} = 1, \cdots, N$:

[A"] For any $\epsilon > 0$ and $\eta > 0$, there is $\delta_{\epsilon\eta}(\tilde{s}) < 1$ such that if $\delta \in (\delta_{\epsilon\eta}(\tilde{s}), 1)$ then in any symmetric equilibrium of the continuation game $\Gamma_{\tilde{s}}(g|\delta)$, (i) all agents will have adopted within $\kappa(\delta)$ periods if their types are below $\tau_{\tilde{s}}(g) - \epsilon$, but would never adopt while $s = \tilde{s}$ if their types exceed $\tau_{\tilde{s}}(g) + \epsilon$, and (ii) for all $t \geq 0$ the equilibrium payoff of t-type is within ϵ of that in the focal equilibrium of $\Gamma_{\tilde{s}}(g)$, i.e., $v^*_{\tilde{s}}(t|g)$.

Note that this property holds trivially for $\tilde{s} = 1$. For induction purposes, we suppose that [A"] holds for all $\tilde{s} < \tilde{r}$ where $\tilde{r} = 2, 3, \cdots, N$. Below we show that it holds for $\tilde{s} = \tilde{r}$ as well. For the proof we need the next two lemmas.

Consider an arbitrary symmetric equilibrium $(a^i)_i$ of the continuation game $\Gamma_{\tilde{r}}(g|\delta)$ with any density g that satisfies (3). Let h^0_z denote a history of no entry up to period $z-1$ of $\Gamma_{\tilde{s}}(g|\delta)$, so that the state in period z is still $\tilde{r}$, where $h^0_1 = \emptyset$. Let $g_1(\delta) = g$ and $g_z(\delta)$, $z = 2, \cdots$, be the posterior in the z-th period that the state has been $\tilde{r}$, i.e., updated from $g_{z-1}(\delta)$ by the adoption probabilities $a^i(\cdot|h^0_{z-1})$. Let $t_z(\delta)$, $z = 1, 2, \cdots$, be the supremum of all types that adopt with a positive probability in the z-th period of state $\tilde{r}$, i.e, $t_z(\delta) = \sup\{t | a^i(t|h^0_z) g_z(\delta)(t) > 0\}$. For notational ease we use g_z and t_z as shorthand for $g_z(\delta)$ and $t_z(\delta)$ when suppressing δ causes no confusion.

Lemma A2: *For any $\zeta > 0$ there is $\delta'_{\tilde{r}}(\zeta) < 1$ such that the following holds if $\delta \in (\delta'_{\tilde{r}}(\zeta), 1)$: for any symmetric equilibrium $(a^i)_i$ of any continuation game $\Gamma_{\tilde{r}}(g|\delta)$,*

$$\sup_z t_z \leq \tau_{\tilde{r}}(g) + \zeta. \tag{17}$$

Proof. To prove by contradiction, suppose otherwise, i.e., there exist some $\zeta > 0$ and a sequence of continuation games $\{\Gamma_{\tilde{r}}(g^\ell|\delta_\ell)\}_\ell$ such that

$$\lim_{\ell\to\infty} \delta_\ell = 1, \ \hat{t}^\ell := \sup_z t_z(\delta_\ell) > \tau_{\tilde{r}}(g^\ell) + \zeta, \text{ and } \hat{t} := \lim_{\ell\to\infty} \hat{t}^\ell \geq \lim_{\ell\to\infty} \tau_{\tilde{r}}(g^\ell) + \zeta. \tag{18}$$

We present our analysis as if $\hat{t}^\ell = t_{z_\ell}(\delta_\ell)$ for some $z_\ell < \infty$ for all ℓ for expositional ease, but an analogous argument applies in the alternative case (because the strategic situation of $\hat{t}^\ell$-type is arbitrarily closely approximated by $t_z(\delta_\ell)$-type as $z \to \infty$). Without loss of generality, we assume that $z_\ell = 1$ for all ℓ, and that ζ is small so that $\tau_{\tilde{r}}(g^\ell)+\zeta < \tau_{\tilde{r}-1}(g^\ell)$ for all ℓ by Lemma A1.

First consider the case that $\hat{t}^\ell < \tau_{\tilde{r}-1}(g^\ell)$ for arbitrarily large ℓ. By taking a subsequence of $\{\Gamma_{\tilde{r}}(g^\ell|\delta_\ell)\}_\ell$ if necessary, we may assume that $\hat{t}^\ell < \tau_{\tilde{r}-1}(g^\ell)$ for all ℓ. Consider the following two strategies in the first period of state $\tilde{r}$:

[a] Adopt in the current period;
[b] Wait until an entry occurs and then adopt in the subsequent period.

By the induction hypothesis [A"], for the purpose of calculating the expected payoff, the strategy [a] entails subsequent adoption process arbitrarily close to the focal equilibrium of $\Gamma_{\tilde{r}-j-1}(g_2^\ell)$ as $\ell \to \infty$, where j is the number of other adopters in the current period. Hence, the expected payoff of a t-type agent from this strategy, denoted by $V_\ell^a(t)$, satisfies

$$\sup_t |V_\ell^a(t) - v_\ell^a(t)| \to 0 \quad \text{as} \quad \ell \to \infty, \quad \text{where}$$
$$v_\ell^a(t) := \sum_{j=0}^{\tilde{r}-1} p_1^\ell(j)\Big((1-\delta_\ell)u_t(N-\tilde{r}+j+1) + \delta_\ell v_{\tilde{r}-j-1}^{in}(t|g_2^\ell)\Big), \tag{19}$$

$p_i^\ell(j)$ is the exante probability that j other agents adopt in the i-th period of state $\tilde{r}$ according to the equilibrium of the continuation game $\Gamma_{\tilde{r}}(g^\ell|\delta_\ell)$, and $v_{\tilde{r}-j-1}^{in}(t|g)$ is the expected payoff of a t-type agent who already has adopted in a certain period with state $s = \tilde{r}-j-1$, when the focal equilibrium ensues in the continuation game $\Gamma_{\tilde{r}-j-1}(g)$.

For an agent of type $t \le \hat{t}^\ell < \tau_{\tilde{r}-1}(g^\ell)$, the strategy [b] also entails, once there is entry, a subsequent adoption process that is arbitrarily close to the focal equilibrium of the continuation game as $\ell \to \infty$. Hence, the expected payoff of a t-type agent from this strategy, denoted by $V_\ell^b(t)$, satisfies

$$\sup_t |V_\ell^b(t) - v_\ell^b(t)| \to 0 \quad \text{as} \quad \ell \to \infty, \quad \text{where}$$
$$v_\ell^b(t) := \sum_{i=1}^{\infty}\sum_{j=1}^{\tilde{r}-1} \delta_\ell^i p_i^\ell(j) v_{\tilde{r}-j}^*(t|g_{i+1}^\ell). \tag{20}$$

Since the same logic used in the proof of Lemma 4 straightforwardly verifies that

$$v_{\tilde{r}-j-1}^{in}(t|g_2^\ell) = v_{\tilde{r}-j}^*(t|g_2^\ell)$$

for $t \le \tau_{\tilde{r}-j}(g_2^\ell)$ and $j \ge 1$, we deduce that

$$\begin{aligned} v_\ell^a(t) - v_\ell^b(t) &= (1-\delta_\ell)\Big(\sum_{j=0}^{\tilde{r}-1} p_1^\ell(j)u_t(N-\tilde{r}+j+1)\Big) \\ &\quad + \delta_\ell p_1^\ell(0) v_{\tilde{r}-1}^{in}(t|g_2^\ell) - \sum_{i=2}^{\infty}\sum_{j=1}^{\tilde{r}-1} \delta_\ell^i p_i^\ell(j) v_{\tilde{r}-j}^*(t|g_{i+1}^\ell). \end{aligned} \tag{21}$$

From the perspective of a certain agent, say 1, that uses the strategy [b], let X_{ij} denote the event that j other agents adopt in the i-th period of state $\tilde{r}$. Consider the hypothetical situation that once there is entry, the focal equilibrium ensues in the continuation game. Then, by the same logic as that in the proof of Lemma 4 again, in the contingency that X_{ij} were to happen for any $i \geq 1$ and $j \geq 1$, the final outcome would be the same regardless of whether the agent 1 uses strategy [a] or [b]. For the contingency that no entry would take place in case the agent 1 uses [b], an event that we denote by Y (which is the complement of $\cup_{i \geq 0, j \geq 1} X_{ij}$), the agent 1 (of type t) would get a payoff of $v^{in}_{\tilde{r}-1}(t|\lim_z g^\ell_z)$ from [a], and 0 from [b]. Therefore, by subtracting the un-discounted expected payoff of [b] from that of [a] in this hypothetical situation, we get

$$p^\ell_1(0) v^{in}_{\tilde{r}-1}(t|g^\ell_2) - \sum_{i=2}^{\infty} \sum_{j=1}^{\tilde{r}-1} p^\ell_i(j) v^*_{\tilde{r}-j}(t|g^\ell_{i+1}) = p_Y v^{in}_{\tilde{r}-1}(t|\lim_z g^\ell_z) \tag{22}$$

where p_Y is the probability of the event Y. From (21) and (22),

$$\begin{aligned} v^a_\ell(t) - v^b_\ell(t) &= (1-\delta_\ell)\Big(\sum_{j=0}^{\tilde{r}-1} p^\ell_1(j) u_t(N - \tilde{r} + j + 1)\Big) \\ &\quad + \delta_\ell \Big(\sum_{i=2}^{\infty} \sum_{j=1}^{\tilde{r}-1} (1 - \delta^{i-1}_\ell) p^\ell_i(j) v^*_{\tilde{r}-j}(t|g^\ell_{i+1}) + p_Y v^{in}_{\tilde{r}-1}(t|\lim_z g^\ell_z)\Big). \end{aligned} \tag{23}$$

The derivatives of $v^*_{\tilde{r}-j}(t|g^\ell_{i+1})$ and $v^{in}_{\tilde{r}-1}(t|\lim_z g^\ell_z)$ with respect to t, being convex combinations of $u_t(n)$'s, are all negative and bounded away from 0 because $|\dot{u}_t(n)| > \theta$ by supposition. From (23), therefore, there is ℓ' and $\theta' > 0$ such that

$$\frac{d(v^a_\ell(t) - v^b_\ell(t))}{dt} < -\theta' \quad \forall t \in (0, \hat{t}) \quad \text{for all} \quad \ell > \ell' \tag{24}$$

because, in particular, $p_Y \geq (1 - F(\bar{t}))^{\tilde{r}-1} > 0$ due to (3).

Since [a] is optimal for $\hat{t}^\ell$-type by supposition, $v^a_\ell(\hat{t}^\ell) - v^b_\ell(\hat{t}^\ell) > -\epsilon$ for sufficiently large ℓ for any given $\epsilon > 0$. By (24), therefore, for any $\epsilon' > 0$, agents of all types $t < \hat{t}^\ell - \epsilon'$ would strictly prefer [a] to [b] for sufficiently large ℓ. If all such types indeed adopt in period $z_\ell = 1$, then we would have $g^\ell_2 \to g^\ell|_{t \geq \hat{t}^\ell}$ as $\ell \to \infty$, which would imply that a $\hat{t}^\ell$-type agent would strictly prefer to wait in period 1 because i) $\hat{t}^\ell > \tau_{\tilde{r}}(g^\ell)$ and ii) a $\tau_{\tilde{r}}(g^\ell)$-type agent (for whom, by definition, $v^a_\ell(\tau_{\tilde{r}}(g^\ell)) - v^b_\ell(\tau_{\tilde{r}}(g^\ell)) \to 0$ as $\ell \to \infty$ if all agents of types $t < \tau_{\tilde{r}}(g^\ell)$ were to adopt in period 1) would strictly prefer to wait if all agents of types $t < \hat{t}^\ell$ were to adopt in period 1. This would be a contradiction to [a] being optimal for $\hat{t}^\ell$-type and thus, would prove (17). This is our core logic of proving Lemma A2, however, the details are more complicated as elaborated below, because we need to take care of the possibility that an agent of types $t < \hat{t}^\ell - \epsilon'$, although strictly prefers [a] to [b] as asserted above, may still find some other strategy optimal, e.g., to wait for several periods (of state $\tilde{r}$) before adopting.

Continuing with the proof, the next step is to show that

$$\lim_{\ell \to \infty} \lim_{z \to \infty} G_z^\ell(\hat{t}^\ell) = 0. \tag{25}$$

To prove this, suppose otherwise, i.e., this limit, which we may assume exists by taking a subsequence if necessary, is strictly positive. Then, since $G_z^\ell(\cdot)$ is continuous with uniformly bounded slope as per (3), there is $t_\ell < \hat{t} - \alpha$ for some $\alpha > 0$ for arbitrarily large ℓ, such that it is optimal for a t_ℓ-type agent to use strategy [b] in the first period of state $\tilde{r}$, i.e., $V_\ell^b(t_\ell) \geq V_\ell^a(t_\ell)$ so that, by (19) and (20), $v_\ell^b(t_\ell) \geq v_\ell^a(t_\ell) - \epsilon$ for sufficiently large ℓ for any small $\epsilon > 0$. Since $v_\ell^a(t_\ell) - v_\ell^b(t_\ell) > v_\ell^a(\hat{t}^\ell) - v_\ell^b(\hat{t}^\ell) + \beta$ for some $\beta > 0$ by (24), $v_\ell^b(\hat{t}^\ell) > v_\ell^b(t_\ell) + \beta - (v_\ell^a(t_\ell) - v_\ell^a(\hat{t}^\ell)) \geq v_\ell^a(t_\ell) - \epsilon + \beta - (v_\ell^a(t_\ell) - v_\ell^a(\hat{t}^\ell)) = -\epsilon + \beta + v_\ell^a(\hat{t}^\ell)$ for sufficiently large ℓ. Thus, $v_\ell^b(\hat{t}^\ell) - v_\ell^a(\hat{t}^\ell) > \beta/2$ when $\epsilon = \beta/2$ and, consequently, $V_\ell^b(\hat{t}^\ell) - V_\ell^a(\hat{t}^\ell) > \beta/4$ for sufficiently large ℓ. This contradicts the optimality of strategy [a] for $\hat{t}^\ell$-type, proving (25).

By (25), for each ℓ one can find $z'_\ell \geq z_\ell = 1$ and $t'_\ell < \hat{t}^\ell$ such that a t'_ℓ-type agent adopts in period z'_ℓ of state $\tilde{r}$ with a positive probability in equilibrium and $G^\ell_{z'_\ell+1}(\hat{t}^\ell) \to 0$ as $\ell \to \infty$. Observe that as of period z'_ℓ, the expected benefit of adopting in period z'_ℓ as opposed to waiting is arbitrarily closely approximated by (23) as $\ell \to \infty$, where $p_i^\ell(j)$ vanishes for all $i \geq 2$ and $j \geq 0$ and $\lim_z g_z^\ell \to g^\ell_{t \geq \hat{t}^\ell}$. Hence, this expected benefit converges to $p_Y v^{in}_{\tilde{r}-1}(t | g^\ell_{t \geq \hat{t}^\ell})$. Recall that $\tau_{\tilde{r}}(g^\ell)$ solves $\Psi(\hat{t} | g, \delta) = 0$ when $\delta = 1$ and $p^+_{\ell,j} = q_{\ell,j}$ for all $\ell \geq 1$ and $j \geq 1$ where Ψ is defined in (15), which implies that $v^{in}_{\tilde{r}-1}(\tau_{\tilde{r}}(g^\ell) | g^\ell_{t \geq \tau_{\tilde{r}}(g^\ell)}) = 0$. Since $\tau_{\tilde{r}}(g^\ell) + \zeta < \hat{t}^\ell$ from (18) and $\dot{u}_t(n) < -\theta$, it further follows that

$$v^{in}_{\tilde{r}-1}(\hat{t}^\ell | g^\ell_{t \geq \hat{t}^\ell}) < -\zeta\theta/2 < 0 \quad \text{for all sufficiently large} \quad \ell. \tag{26}$$

Consequently, for a t'_ℓ-type agent to prefer to adopt in period z'_ℓ for sufficiently large ℓ (i.e., for $\lim_{\ell \to \infty} p_Y v^{in}_{\tilde{r}-1}(t'_\ell | g^\ell_{t \geq \hat{t}^\ell}) \geq 0$), therefore,

$$t'_\ell < \hat{t}^\ell - \lambda' \text{ for all sufficiently large } \ell \text{ for some } \lambda' > 0. \tag{27}$$

The equilibrium strategy of a t'_ℓ-type agent, denoted by [c], is: to follow [b] until the period z'_ℓ, then adopt in the period z'_ℓ. By the same reasoning as before, the expected payoff of t-type agent from the strategy [c], denoted by $V_\ell^c(t)$, satisfies

$$\sup_t |V_\ell^c(t) - v_\ell^c(t)| \to 0 \quad \text{as} \quad \ell \to \infty, \quad \text{where}$$

$$\begin{aligned} v_\ell^c(t) := & \sum_{i=1}^{z'_\ell - 1} \sum_{j=1}^{\tilde{r}-1} \delta_\ell^i p_i^\ell(j) v^*_{\tilde{r}-j}(t | g^\ell_{i+1}) \\ & + \delta_\ell^{z'_\ell - 1} \sum_{j=0}^{\tilde{r}-1} p^\ell_{z'_\ell}(j) \Big((1 - \delta_\ell) u_t(N - \tilde{r} + j + 1) + \delta_\ell v^{in}_{\tilde{r}-j-1}(t | g^\ell_{z'_\ell + 1}) \Big). \end{aligned} \tag{28}$$

Since $v^{in}_{\tilde{r}-j-1}(t|g^{\ell}_{z'_\ell+1}) = v^{*}_{\tilde{r}-j}(t|g^{\ell}_{z'_\ell+1})$ by the same logic of the proof of Lemma 4, and

$$p^{\ell}_1(0)v^{in}_{\tilde{r}-1}(t|g^{\ell}_2) - \sum_{i=2}^{z'_\ell}\sum_{j=1}^{\tilde{r}-1} p^{\ell}_i(j)v^{*}_{\tilde{r}-j}(t|g^{\ell}_{i+1}) = p^{\ell}_{z'_\ell}(0)v^{in}_{\tilde{r}-1}(t|g^{\ell}_{z'_\ell+1})$$

by the same reasoning as that that led to (22), from (19) and (28) we calculate that

$$\begin{aligned} v^a_\ell(t) - v^c_\ell(t) \; &= (1-\delta_\ell)\Big(\sum_{j=0}^{\tilde{r}-1}(p^{\ell}_1(j) - \delta_\ell^{z'_\ell-1}p^{\ell}_{z'_\ell}(j))u_t(N-\tilde{r}+j+1)\Big) \\ &\quad + \delta_\ell \sum_{i=2}^{z'_\ell-1}\sum_{j=1}^{\tilde{r}-1}(1-\delta_\ell^{i-1})p^{\ell}_i(j)v^{*}_{\tilde{r}-j}(t|g^{\ell}_{i+1}) \\ &\quad + \delta_\ell \sum_{j=0}^{\tilde{r}-1}(1-\delta_\ell^{z'_\ell-1})p^{\ell}_{z'_\ell}(j)v^{in}_{\tilde{r}-j-1}(t|g^{\ell}_{z'_\ell+1}). \end{aligned} \tag{29}$$

Since the derivatives of $v^{*}_{\tilde{r}-j}(t|g^{\ell}_{i+1})$ and $v^{in}_{\tilde{r}-j-1}(t|g^{\ell}_{z'_\ell+1})$ with respect to t are all negative and bounded away from 0 as argued earlier, for any given $\epsilon > 0$, the derivatives of $v^a_\ell(t) - v^c_\ell(t)$ with respect to t is lower than ϵ uniformly if ℓ is sufficiently large according to (29). Since, in addition, $v^a_\ell(t'_\ell) - v^c_\ell(t'_\ell) \to \gamma \le 0$ as $\ell \to \infty$ because [c] is optimal for t'_ℓ-type, for [a] to be optimal for the $\hat{t}^\ell$-type it must follow that

$$v^a_\ell(\hat{t}^\ell) - v^c_\ell(\hat{t}^\ell) \to 0 \quad \text{as} \quad \ell \to \infty. \tag{30}$$

Observe further that, since the strategies [b] and [c] lead to final networks of the same size as long as some other agent adopts in period z'_ℓ or earlier and the payoff from [b] in the alternative contingency is non-negative, for any $\epsilon > 0$ the following should hold for all sufficiently large ℓ:

$$v^b_\ell(t) - v^c_\ell(t) \ge -\delta_\ell^{z'_\ell}p^{\ell}_{z'_\ell}(0)\Big((1-\delta_\ell)u_t(N-\tilde{r}+1) + \delta_\ell v^{in}_{\tilde{r}-1}(t|g^{\ell}_{z'_\ell+1})\Big) - \epsilon. \tag{31}$$

If $\delta_\ell^{z'_\ell} \to 0$ as $\ell \to \infty$, therefore, $v^b_\ell(t) - v^c_\ell(t) > -\epsilon$ for sufficiently large ℓ for any $\epsilon > 0$ and thus, $V^b_\ell(t) - V^c_\ell(t) > -\epsilon$ for sufficiently large ℓ by (20) and (28). This, together with $V^a_\ell(t'_\ell) - V^c_\ell(t'_\ell) \le 0$, would imply that $V^a_\ell(t'_\ell) - V^b_\ell(t'_\ell) < \epsilon$ for any $\epsilon > 0$ if ℓ is sufficiently large. Then, since $\lim_{\ell\to\infty} t'_\ell$, which we may assume exists by taking a subsequence if necessary, is lower than $\hat{t}$ at least by $\lambda' > 0$ by (27), it would follow from (24) that $V^a_\ell(\hat{t}_\ell) - V^b_\ell(\hat{t}_\ell) < 0$ for sufficiently large ℓ, a contradiction to [a] being optimal for a $\hat{t}^\ell$-type agent. Therefore, we must have $\delta_\ell^{z'_\ell} \to \gamma' > 0$ as $\ell \to \infty$.

Then, since the RHS of (31) is positive and bounded away from 0 at $t = \hat{t}^\ell$ as $\ell \to \infty$ and $\epsilon \to 0$ by (25) and (26), it follows that $v^b_\ell(\hat{t}^\ell) > v^c_\ell(\hat{t}^\ell)$ for sufficiently large ℓ. Together with (30), this would imply that [a] is not optimal for a $\hat{t}^\ell$-type in period z_ℓ, a contradiction. This completes the proof of Lemma A2 for the case that $\hat{t}^\ell < \tau_{\tilde{r}-1}(g^\ell)$ for arbitrarily large ℓ.

In the alternative case that $\hat{t}^\ell \geq \tau_{\tilde{r}-1}(g^\ell)$ for all sufficiently large ℓ, the same argument applies with the straightforward modification: [b] describes a strategy that an agent adopts only when a state is reached such that adopting is the equilibrium strategy in the continuation game. This entails more complex algebraic formaulae but the core logic of the proof is the same. Hence, the details are omitted here. $\square$

By the induction hypothesis [A"], by adopting in the first period of the continuation game $\Gamma_{\tilde{r}}(g|\delta)$, an agent of a type $t < \tau_{\tilde{r}}(g)$ can guarantee himself a payoff arbitrarily close to that from the focal equilibrium, $v^*_{\tilde{r}}(t|g)$, as $\delta \to 1$. Since $v^*_{\tilde{r}}(t|g)$ is bounded (uniformly across t and g) and is arbitrarily close to an upper bound of the maximum possible equilibrium payoff for t-type as $\delta \to 1$ by Lemma A2, for any $\epsilon > 0$ there is $\delta''(\epsilon)$ such that if $\delta > \delta''(\epsilon)$ then the equilibrium payoff of t-type is within ϵ of $v^*_{\tilde{r}}(t|g)$ for all g and $t < \tau_{\tilde{r}}(g)$.

Lemma A3: *For any $\epsilon > 0$ and $\eta > 0$, there is $\delta_{\epsilon\eta}(\tilde{r}) < 1$ such that if $\delta \in (\delta_{\epsilon\eta}(\tilde{r}), 1)$ then in the continuation game $\Gamma_{\tilde{r}}(g|\delta)$ an agent of any type below $\tau_{\tilde{r}}(g) - \epsilon$ will have adopted within $\kappa(\delta)$ periods where $\kappa(\delta)$ is the largest integer satisfying $\delta^{\kappa(\delta)} > 1 - \eta$.*

Proof. Suppose otherwise, i.e., there exist sequences $\{\delta_\ell\}$, $\{g^\ell\}$ and $\{t^\ell\}$ such that $\lim_{\ell\to\infty} \delta_\ell = 1$, and for all $\ell \geq 1$, $t^\ell \leq \tau_{\tilde{r}}(g^\ell) - \epsilon$ and a t^ℓ-type agent would not adopt within $\kappa(\delta_\ell)$ periods of state $\tilde{r}$ in an equilibrium of $\Gamma_{\tilde{r}}(g^\ell|\delta_\ell)$.

For any $\zeta > 0$, for ℓ sufficiently large, no agent of type greater than $\tau_{\tilde{r}}(g^\ell) + \zeta$ adopts while $s = \tilde{r}$ by Lemma A2. By taking subsequences if necessary, therefore, we may assume for each $\ell \geq 1$ that no agent of type greater than $\tau_{\tilde{r}}(g^\ell) + 1/\ell$ adopts while $s = \tilde{r}$.

Let Y_ℓ be the event that all other (remaining) agents have types greater than $\tau_{\tilde{r}}(g^\ell) + 1/\ell$, from the perspective of a t^ℓ-type agent in $\Gamma_{\tilde{r}}(g^\ell|\delta_\ell)$. In the complement event, Y^c_ℓ, the maximum possible equilibrium payoff of this agent is arbitrarily closely approximated by that obtainable when all agents of types less than $\tau_{\tilde{r}}(g^\ell) + 1/\ell$ adopt in the first period, followed by the focal continuation equilibrium. Let $\bar{V}_\ell$ denote the expected payoff of this agent when all agents (including himself) behave like this, conditional on the event Y^c_ℓ. If a t^ℓ-type agent followed the supposed equilibrium strategy of not adopting in the first $\kappa(\delta_\ell)$ periods of state $\tilde{r}$, his expected payoff conditional on the event Y^c_ℓ is bounded above by an upper bound arbitrarily closely approximated by $\bar{V}_\ell$. If a t^ℓ-type agent followed the same strategy, conditional on the event Y_ℓ, no agent would adopt in the first $\kappa(\delta_\ell)$ periods due to Lemma A2. In the subsequent period, since $t^\ell < \tau_{\tilde{r}}(g^\ell|_{t \geq \tau_{\tilde{r}}(g^\ell)+1/\ell})$ for sufficiently large ℓ by continuity because $\tau_{\tilde{r}}(g^\ell|_{t \geq \tau_{\tilde{r}}(g^\ell)}) = \tau_{\tilde{r}}(g^\ell) > t^\ell$ by [A] (ii) of Section 3, a t^ℓ-type agent obtains a positive payoff by adopting: In fact, the maximum possible payoff obtainable in the continuation game is bounded above by an upper bound arbitrarily closely approximated by the payoff obtainable when he adopts in the current period, followed by the focal continuation equilibrium. Let $\bar{V}'_\ell$ denote the expected payoff of this agent when all agents behave like this, conditional on the event Y_ℓ.

The payoff of a t^ℓ-type agent following the supposed equilibrium strategy of not adopting in the first $\kappa(\delta_\ell)$ periods of state $\tilde{r}$, therefore, is bounded above by an upper bound arbitrarily closely approximated by $(1-p_{Y_\ell})\bar{V}_\ell + (1-\eta)p_{Y_\ell}\bar{V}'_\ell$, where p_{Y_ℓ} is the probability of event Y_ℓ, which in turn is arbitrarily closely approximated by

$$v^*_{\tilde{r}}(t^\ell|g^\ell) - \eta\Big(1 - \int_0^{\tau_{\tilde{r}}(g^\ell)} g^\ell dt\Big)^{\tilde{r}-1} \cdot v^*_{\tilde{r}}(t^\ell|g^\ell|_{t\geq\tau_{\tilde{r}}(g^\ell)}) < v^*_{\tilde{r}}(t^\ell|g^\ell) - \eta\big(1-F(\bar{t})\big)^{\tilde{r}-1}\epsilon\theta$$

as $\ell \to \infty$ because (i) $(1-p_{Y_\ell})\bar{V}_\ell + p_{Y_\ell}\bar{V}'_\ell$ converges to $v^*_{\tilde{r}}(t^\ell|g^\ell)$ as $\ell \to \infty$ by definition, (ii) $v^*_{\tilde{r}}(t^\ell|g^\ell|_{t\geq\tau_{\tilde{r}}(g^\ell)}) - \epsilon\theta > v^*_{\tilde{r}}(\tau_{\tilde{r}}(g^\ell)|g^\ell|_{t\geq\tau_{\tilde{r}}(g^\ell)}) = 0$ since $t^\ell < \tau_{\tilde{r}}(g^\ell) - \epsilon$ and $|\dot{u}_t(n)| > \theta$ for all $t \geq 0$ and $n \geq 1$, and (iii) $\int_0^{\tau_{\tilde{r}}(g^\ell)} g^\ell dt < F(\bar{t})$ by (3). This contradicts the earlier observation that, for any $\epsilon' > 0$, the equilibrium payoff of a t-type is within ϵ' of $v^*_{\tilde{r}}(t|g)$ for all g and $t < \tau_{\tilde{r}}(g)$ as $\delta \to 1$. □

Combining Lemmas A2 and A3, we have proved that the property (i) of [A"] holds for $s = \tilde{r}$ as well. Recall that we already proved property [A"] (ii) for $t < \tau_{\tilde{r}}(g)$ before Lemma A3. Observe that the property [A"] (i) ensures that, from any agent's perspective, all other agents behave arbitrarily closely to the strategy of the focal equilibrium as $\delta \to 1$. Therefore, for an agent whose type is $t \in (\tau_{\tilde{r}}(g), \tau_{\tilde{r}-1}(g))$, by employing the strategy of adopting in the first period that the state has turned to $s \leq \tilde{r}-1$, by continuity, warrants an expected payoff that is arbitrarily close to $v^*_{\tilde{r}}(t|g)$ as $\delta \to 1$. Since the aforementioned strategy is optimal in the focal equilibrium for the considered types, again by continuity, no other strategy would generate an expected payoff that exceeds $v^*_{\tilde{r}}(t|g) + \epsilon$ as $\delta \to 1$ for any $\epsilon > 0$, proving the property [A"] (ii) for $t \in (\tau_{\tilde{r}}(g), \tau_{\tilde{r}-1}(g))$. By an analogous argument one can prove the property [A"] (ii) for $t \in (\tau_{\tilde{r}-\iota}(g), \tau_{\tilde{r}-\iota-1}(g))$ for each $\iota = 1, \cdots, \tilde{r}-2$. This completes the proof. ■

Proof of Theorem 3. Let τ_s and $\tilde{\tau}_s$ denote the equilibrium cutoff levels in state s when the population size is N and $N+1$, respectively. Then, $\tau_{n|N} = \tau_{N-n}$ and $\tau_{n|N+1} = \tilde{\tau}_{N-n+1}$. Hence, the proof amounts to showing that $\tau_s < \tilde{\tau}_{s+1}$ for all $s = 1, \cdots, N$.

Clearly, $\tau_1 < \tilde{\tau}_1$ because a τ_1-type ($\tilde{\tau}_1$-type, resp) agent derives a utility of 0 from being a member of a network of size N ($N+1$, resp). Suppose $N-1$ agents already joined and two agents remain. A remaining agent of the cutoff type, $\tilde{\tau}_2$, must have an expected utility of 0 conditional on being the sole adopter in this period due to Lemma 4, which cannot be the case if $\tilde{\tau}_2 \leq \tau_1$ (because the final network size will be at least N, and higher with a positive probability). Hence, $\tau_1 < \tilde{\tau}_2$.

For induction purposes, suppose $\tau_s < \tilde{\tau}_{s+1}$ for all $s < r$ where $r = 2, \cdots, N$. We now show that $\tau_r < \tilde{\tau}_{r+1}$. Recall from the proof of Theorem 1 that a τ_r-type agent has an expected utility of 0 conditional on him being the only adopter in a period with r remaining agents after $N-r$ had already joined. Given this, suppose $N-r$ agents already joined and $r+1$ agents remain in some period k. If the cutoff level, $\tilde{\tau}_{r+1}$, was equal to τ_r, conditional on there being only one adopter in period k, the distribution of the eventual additional adopters first-order stochastically dominates that in the case when r agents remained, because one more agent remains and $\tau_s < \tilde{\tau}_{s+1}$ for all $s < r$ by induction hypothesis. Hence, the expected utility of a $\tilde{\tau}_{r+1}$-type agent conditional on being the sole adopter in this period, which should be 0 in equilibrium, would instead be strictly positive, a contradiction. If $\tilde{\tau}_{r+1} < \tau_r$, a similar argument would lead to the same contradiction. Therefore, we conclude that $\tau_r < \tilde{\tau}_{r+1}$, as desired. ■

Proof of Theorem 4. (a) Since $\tau_{0|N}$ increases in N by Theorem 3, the sequence $\{\tau_{0|N}\}_N$ either explodes or converges. If $\tau_{0|N} \to \infty$ as $N \to \infty$, then $F(\tau_{0|N}) \to 1$. Furthermore, $F(\bar{t}_N) \to 1$ as well since $\tau_{0|N} \leq \bar{t}_N$ for all N. Thus, $\frac{F(\tau_{0|N})}{F(\bar{t}_N)} \to 1$ follows. Suppose otherwise, i.e., $\tau_{0|N} \to \tau_{0|\infty} < \infty$. Let $\lim_{N\to\infty} \bar{t}_N = \bar{t}_\infty$, with $\bar{t}_\infty = \infty$ allowed. If $F(\bar{t}_\infty) = F(\tau_{0|\infty})$, the claim holds trivially. Hence, suppose otherwise, i.e., $F(\bar{t}_\infty) > F(\tau_{0|\infty})$. Note that $\tau_{n|\infty} := \lim_{N\to\infty} \tau_{n|N}$ is well-defined for each $n = 1, \cdots$. Since $\tau_{n|\infty} \leq \tau_{n+1|\infty}$ and $\lim_{n\to\infty} \tau_{n|\infty} = \bar{t}_\infty$, there exist $\gamma > 0$ and an integer $n' \geq 1$ such that $F(\tau_{n'-1|\infty}) + \gamma < F(\tau_{n'|\infty})$. Then, in the contingency that an agent adopted alone in a period after $n'-1$ other agents have adopted previously, the expected number of agents who would adopt in the next period exceeds $\gamma(N-n')/2$ for sufficiently large N. Consequently, the expected utility of the $\tau_{n'-1|N}$-type agent conditional on adopting alone in a period of state $N-n'+1$, which should be 0 in equilibrium, would instead exceed $u_{\tau_{n'-1|N}}(n' + \gamma(N-n')/2)$ for sufficiently large N. This is a contradiction because $\lim_{N\to\infty} u_{\tau_{n'-1|N}}(n' + \gamma(N-n')/2) = \lim_{N\to\infty} u_{\tau_{n'-1|\infty}}(N) > 0$ where the inequality follows from $\tau_{n'-1|\infty} < \bar{t}_\infty$ (which is implied by $F(\tau_{n'-1|\infty}) + \gamma < F(\tau_{n'|\infty})$), completing the proof of (a).

(b) Fix an arbitrary sequence $\{(a^i_N)_i\}_N$. The logic of Lemma 4 and the subsequent argument leading to the equilibrium being a cutoff strategy equilibrium when $\delta = 1$, extend straightforwardly to asymmetric equilibria (hence, the details are omitted here). Therefore, a^i_N is a cutoff strategy for each agent $1 \leq i \leq N$ and $N = 2, \cdots$. Consequently, $\tau^i_{0|N}$ is the agent i's initial cutoff level and $\tau^\chi_{0|N}$ is the lowest among those that belong to the fraction χ of the highest initial cutoff levels of all agents in the equilibrium profile $(a^i_N)_i$.

Fix any $\chi < 1$. If $\tau^\chi_{0|N} \to \infty$ as $N \to \infty$, then $F(\tau^\chi_{0|N}) \to 1$ and $\frac{F(\tau^\chi_{0|N})}{F(\bar{t}_N)} \to 1$ follow for the same reason as before. Suppose otherwise. Then, there exists a subsequence $\{N_j\}_{j=1}^\infty$ of $\{1, 2, \cdots\}$ such that $\tau^\chi_{0|N_j} \to \tau^\chi_{0|\infty} < \infty$ as $j \to \infty$. Since $\bar{t}_N$ increases in N, let $\lim_{N\to\infty} \bar{t}_N = \bar{t}_\infty \geq \tau^\chi_{0|\infty}$, with $\bar{t}_\infty = \infty$ allowed. Note that $\lim_{j\to\infty} \bar{t}_{N_j} = \bar{t}_\infty$ as well.

To prove that $F(\tau^\chi_{0|\infty}) = F(\bar{t}_\infty)$ by contradiction, suppose otherwise, i.e., $F(\tau^\chi_{0|\infty}) <$

$F(\bar{t}_\infty)$. Then, there exist $\gamma > 0$ and an integer $n \geq 0$ such that

$$\tau^{\chi}_{0|\infty} + \gamma < \bar{t}_{n+1} < \bar{t}_\infty \quad \text{and} \quad F(\tau^{\chi}_{0|\infty}) + \gamma < F(\bar{t}_{n+1}). \tag{32}$$

Note that $u_{\tau^i_{n|N}}(n+1) \leq 0$ for all i, because the expected utility of agent i of $\tau^i_{n|N}$-type, conditional on being a sole adopter in a period of state $N-n$, is 0 (due to Lemma 4). Since $u_{\bar{t}_{n+1}}(n+1) = 0$ by definition, it follows that $\bar{t}_{n+1} \leq \tau^i_{n|N}$ for all i and for all $N > n$ and thus, in conjunction with (32),

$$\tau^{\chi}_{0|\infty} + \gamma \, < \, \bar{t}_{n+1} \, \leq \, \tau^i_{n|N_j} \quad \forall i = 1, \cdots, N_j, \quad \forall N_j > n.$$

For each N_j sufficiently large, let $\chi^c(N_j)$ be the set of agents i such that $\tau^i_{0|N_j} \leq \tau^{\chi}_{0|N_j}$. For each $i \in \chi^c(N_j)$, let $\nu^i \in \arg\max_\nu \big(F(\tau^i_{\nu|N_j}) - F(\tau^i_{\nu-1|N_j})\big)$ among all $\nu \in \{1, \cdots, n\}$ such that $\tau^i_{\nu-1|N_j} \leq \bar{t}_{n+1}$; Let $\hat{\nu}(N_j) \in \{1, \cdots, n\}$ be the most common ν^i among agents in $\chi^c(N_j)$; Let $\hat{\chi}^c(N_j) := \{i \in \chi^c(N_j) | \nu^i = \hat{\nu}(N_j)\}$. It follows from definitions that (i) $\#\chi^c(N_j) \geq N_j(1-\chi)$ and $\#\hat{\chi}^c(N_j) \geq N_j(1-\chi)/n$ where $\#X$ denotes the cardinality of a set X, and (ii) $\min\{F(\tau^i_{\hat{\nu}(N_j)|N_j}) - F(\tau^i_{\hat{\nu}(N_j)-1|N_j}) | i \in \hat{\chi}^c(N_j)\} \geq \gamma/(n+1)$ for all sufficiently large j. By taking a subsequence of $\{N_j\}$ if necessary, assume that $\hat{\nu}(N_j) = \hat{\nu} \in \{1, \cdots, n\}$ for all $j \geq 1$. By reindexing agents if necessary, assume that agent 1 belongs to $\hat{\chi}^c(N_j)$ for all j sufficiently large.

Consider the contingency that agent 1 adopted alone in a period, say k, after $\hat{\nu}-1$ other agents have adopted previously. Since at least $\#\hat{\chi}^c(N_j) - \hat{\nu}$ of the agents in $\hat{\chi}^c(N_j)$ remain in the next period, and each of these agents will adopt in the next period with a probability no less than $\gamma/(n+1)$ as per (ii) above, the expected number of agents who would adopt in the next period exceeds $\gamma(N_j(1-\chi)/n - \hat{\nu})/(n+1)$ according to (i) above for sufficiently large j. Since $\gamma(N_j(1-\chi)/n - \hat{\nu})/(n+1) \to \infty$ as $j \to \infty$, therefore, the expected utility of agent 1 of type $\tau^i_{\hat{\nu}-1|N_j}$ conditional upon adopting alone in a period of state $N_j - \hat{\nu} + 1$, which should be 0 in equilibrium, would instead converge to $\lim_{N\to\infty} u_{\tau^i_{\hat{\nu}-1|\infty}}(N) > 0$ as $j \to \infty$, where the inequality follows because $\tau^i_{\hat{\nu}-1|\infty} < \bar{t}_{n+1}$ by construction.

This contradiction necessitates the conclusion that if $\tau^{\chi}_{0|N_j} \to \tau^{\chi}_{0|\infty} < \infty$ as $j \to \infty$, then $\bar{t}_N \to \bar{t}_\infty$ as $N \to \infty$ where $F(\tau^{\chi}_{0|\infty}) = F(\bar{t}_\infty)$. From this, it further follows that if $\tau^{\chi}_{0|N_j} \to \tau^{\chi}_{0|\infty} < \infty$ as $j \to \infty$ then for any $\epsilon > 0$ there is N_ϵ such that $|F(\tau^{\chi}_{0|N}) - F(\bar{t}_\infty)| < \epsilon$ if $N > N_\epsilon$: If otherwise, there would be a convergent subsequence of $\{\tau^{\chi}_{0|N}\}$ with a limit, say τ', such that $F(\tau') \neq F(\bar{t}_\infty)$, contrary to what we have established above. Hence, the conclusion of part (b) ensues. ∎

Proof of Theorem 5. Since $\tau_1 = \tilde{\tau}_1$, $F(\tau_1) > \tilde{F}(\tilde{\tau}_1)$ is immediate from stochastic dominance of $\tilde{F}$. For induction purposes, suppose $F(\tau_s) > \tilde{F}(\tilde{\tau}_s)$ for all $s < r$ where $r = 2, \cdots, N$. We now show that $F(\tau_r) > \tilde{F}(\tilde{\tau}_r)$. Recall from the proof of Theorem 1 that a τ_r-type agent has an expected payoff of 0 conditional on him being the only adopter in a period with r remaining agents under F. To reach a contradiction, suppose $F(\tau_r) \leq \tilde{F}(\tilde{\tau}_r)$. Consider a $\tilde{\tau}_r$-type agent when r agents remain under $\tilde{F}$: conditional on him being the only adopter in the current period, the distribution of future adoption is first-order stochastically dominated by that under the same condition with $\tilde{F}$ replaced by F, because for any $s < r$, the posterior probability that any remaining agent will join in any period when or before the state have reached s is lower under $\tilde{F}$ than under F by induction hypothesis. Since $\tau_r < \tilde{\tau}_r$ when $F(\tau_r) \leq \tilde{F}(\tilde{\tau}_r)$, this implies that a $\tilde{\tau}_r$-type agent expects a negative discounted future payoff conditional on him being the only adopter among r remaining agents under $\tilde{F}$, hence he would strictly prefer not joining, contrary to our supposition. Therefore, we conclude that $F(\tau_r) > \tilde{F}(\tilde{\tau}_r)$ as desired. ∎

References

[1] A. Admati, M. Perry, Joint Projects without Commitment, Rev. Econ. Stud. 58 (1991), 259–276.

[2] M. Bagnoli, B. Lipman, Provision of public goods: Fully implementing the Core through private contributions, Rev. Econ. Stud. 56 (1989), 583–601.

[3] C. Bliss, B. Nalebuff, Dragon-slaying and ballroom dancing: The private supply of a public good, J. Public Econ. 25 (1984), 1–12.

[4] H. Carlsson, E. van Damme, Global Games and Equilibrium Selection, Econometrica 61 (1993), 989–1018.

[5] C. Chamley, Coordinating Regime Switches, Quart. J. Econ. 114 (1999), 869–905.

[6] J. Choi, Herd behavior, the penguin effect, and the suppression of informational diffusion: an analysis of informational externalities and payoff interdependency, RAND J. Econ. 28 (1997), 407–425.

[7] A. Dasgupta, Coordination and Delay in Global Games, J. Econ. Theory 134 (2007), 195–225.

[8] A. Dixit, Clubs with Entrapment, Amer. Econ. Rev., 93 (2003), 1824–1829.

[9] P. Dybvig, C. Spatt, Adoption externalities as public goods, J. Public Econ. 20 (1983), 231–247.

[10] J. Farrell, G. Saloner, Standardization, Compatibility, and Innovation, RAND J. Econ. 16 (1985), 70–83.

[11] J. Farrell, G. Saloner, Coordination Through Committees and Markets, RAND J. Econ. 19 (1988), 235–252.

[12] D. Gale, Dynamic coordination games, Econ. Theory 5 (1995), 1–18.

[13] D. Gale, Monotone games with positive spillovers, Games Econ. Behav. 37 (2001), 295–320.

[14] P. Heidhues, N. Melissas, Euilibria in a dynamic global game: the role of cohort effects, Econ. Theory 28 (2006), 531–557.

[15] C. Hellwig, A. Mukherji, A. Tsyvinski, Self-Fulfilling Currency Crises: The Role of Interest Rates, Amer. Econ. Rev. 96 (2006), 1769–1787.

[16] L. Marx, S. Matthews, Dynamic Voluntary Contribution to a Public Project, Rev. Econ. Stud. 67 (2000), 327–358.

[17] S. Morris, H.S. Shin, Global Games: Theory and Applications, in: M. Dewatripont, L. Hansen, S. Turnovsky (Eds.), Advances in Econ. and Econometrics Cambridge University Press, Cambridge, 2003, pp. 56–114.

[18] M. Ostrovsky, M. Schwarz, Adoption of standards under uncertainty, RAND J. Econ., 36 (2005), 816–832.

[19] I.-U. Park, A Simple Inducement Scheme to Overcome Adoption Externalities, Contributions to Theoretical Econ. 4 (2004), No. 1, Article 3.

[20] J. Rohlfs, A theory of interdependent demand for a communications service, Bell J. Econ. 5 (1974), 16–37.

[21] L. Simon, M. Stinchcombe, Extensive form games in continuous time: Pure strategies, Econometrica, 57 (1989), 1171–1214.